



Overview (1): Logistics

Total C.F.U.: $9 = (8+1)$ i.e. 76 hr = $(64+12)$ hr

Lectures @ AULA I

on Tuesdays (11 – 13)

Wednesdays (09 – 11)

Fridays (09 – 12)

Problems from tests ~ @ mid of October onwards

NEW

“In itinere” tests (~end of October, ~end of November, ~ mid December

Contact: daniele.dallacasa@unibo.it ddallaca@ira.inaf.it

on (tuesdays) & thursdays 15:30 – 17:00

- • • Best: send an e-mail to agree a meeting, • • • also out of official time slot
- Please, use your academic mail address name.surname@studio.unibo.it



Overview (2): (updated) slides are available at:



<https://people.ira.inaf.it/ddallaca/P-RAD.html>

ATTENZIONE:

- QUANTO DISPONIBILE SUL SITO WEB E' COPIA DELLE SLIDES
- QUESTE POSSONO ESSERE LEGGERMENTE MODIFICATE O AGGIORNATE PRIMA DELLE LEZIONI. IL CONTENUTO RIMANE INVARIATO.
- **NON SONO DISPENSE** & **NON SONO SUFFICIENTI** PER PREPARARE ADEGUATAMENTE L'ESAME



FYI,
Never asked
@ exams

FYI,
not for study

Mostly computations & algebra,
possibly commented @ lectures

Residual stuff from earlier AY.
Now dropped, just ignore!



copy of the slides are available at

Changed!

<https://people.ira.inaf.it/ddallaca/P-RAD.html>

they are just lecture notes, not sufficient for a proper study

Clarke & Carswell – "Astrophysical Fluid Dynamics"

Padmanaban – "Theoretical Astrophysics" (Vol I – Astrophysical processes)

Rybicki & Lightman – "Radiative Processes in Astrophysics"

Longair – "High Energy Astrophysics"

Ghisellini – "Radiative Processes in High Energy Astrophysics"

Dopita & Sutherland – "Astrophysics of the Diffuse Universe"

Tennison – "Astrophysical Spectroscopy"

<http://www.ira.inaf.it/Library/e-books/Fanti&Fanti-Papers.pdf>



Overview (3): Information (cont'd)

Exam:

A. Written admission test: January – February (winter)
June-July (summer), August – September (fall)
+ two more “unofficial” tests (November & April)

[?] Duration: 1.5 hr time

books, (paper) lecture notes, etc. can be used

[?] What is the “short” admission test? Example ➡

A (1). In itinere tests can bring to B

B. Oral: about 1⁺ date/month

3 questions spanning the 3 sections of the course

Nome	Cognome

→ Utilizzabile per un orale entro il 19 Dicembre 2024

1. Si consideri Rigel (β Orioini), avente raggio medio 80 volte quello del sole ($R_{\text{SUN}} = 700000$ km), temperatura fotosferica di circa 12000 K. Si trova ad una distanza di circa 260 pc. Calcolare

A) Il flusso **in unita' solari**, misurato a 30 GHz [0.3]

B) Il flusso **in Jy**, misurato a 30 GHz [0.3]

C) La luminosita' **monocromatica** a 30 GHz [0.25]

D) Le dimensioni angolari in cielo. [0.15]

2. Valutare la validita' delle seguenti affermazioni giustificando la risposta (eventualmente correggendo l'affermazione, qualora ritenuta falsa/inesatta, o scrivendo una frase o esempio di supporto qualora la frase sia ritenuta vera)

A: Il coefficiente di assorbimento nell'equazione del trasporto radiativo, puo' dipendere fortemente dalla λ .

B: Se aumentiamo la temperatura di un plasma termico, aumenta anche la sua luminosita' bolometrica.

C: I meccanismi di accelerazione delle particelle cariche possono essere legati alla presenza di onde d'urto;

D: Siamo in grado di determinarne sempre il luogo di origine dei raggi cosmici

F: L'emissione di sincrotrone da elettroni ultrarelativistici in un campo magnetico ordinato e' polarizzata linearmente

G: Il “colore” di una stella dipende dalla sua luminosita': stelle luminose sono blu, stelle poco luminose sono rosse.

H: Piccole perturbazioni in un fluido magnetizzato non compressibile possono propagarsi con una velocita' caratteristica.

I: Il campo di radiazione Urad e' determinante per stabilire se l'H nello stato 2p puo' effettuare un processo di LASER.

J: Il colore delle galassie ellittiche e' rosso, poiche' la polvere presente in esse assorbe gran parte della luce blu.

K: L'espansione adiabatica di un plasma magnetizzato relativistico fa diminuire drasticamente l'emissione di sincrotrone.

L: Lo scattering Compton aumenta la frequenza della radiazione diffusa rispetto a quella incidente.

A bunch of tests is available @ http://www.ira.inaf.it/~ddallaca/testo_esercizi.pdf

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Overview (4):

Aim(s) of the course:

We will have a
PANCHROMATIC APPROACH
to Astrophysics

Lectures will provide
all (most of) the tools to understand what
we observe in the Universe (i.e. radiation)

Disclaimer: A number of details are omitted or summarized since they would require a lot of time, much more than that allocated to this course.

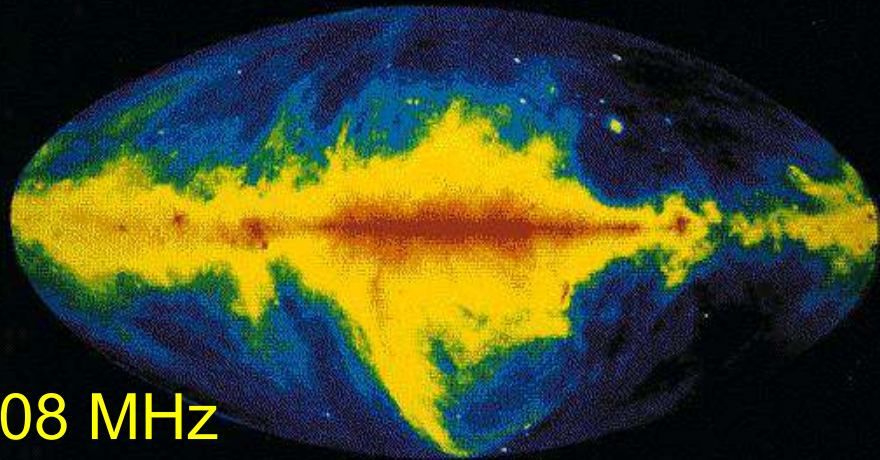


The reason of the "PANCHROMATIC APPROACH"

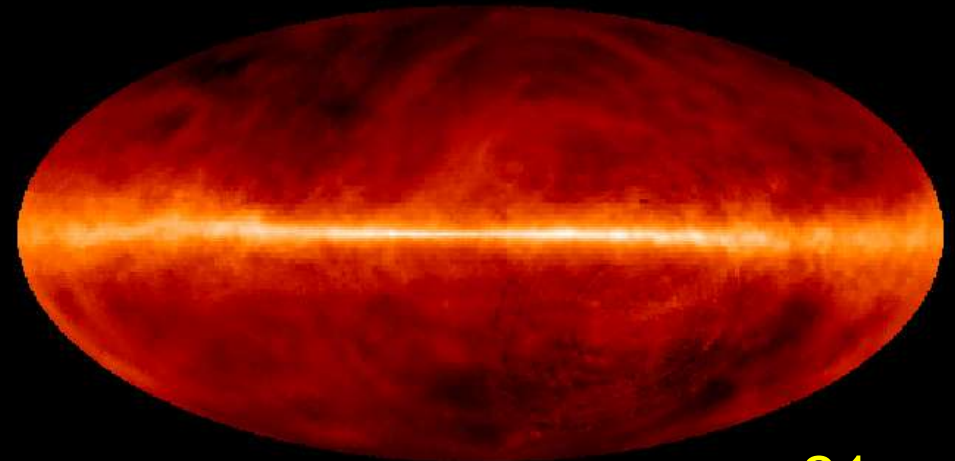
The full Sky (i.e. the Universe) at various wavelengths.

At the end of the course it will be clear how to interpret them (synchrotron, "forbidden" line emission, black body and dust absorption, cosmic rays and matter distribution in the milky way)

408 MHz

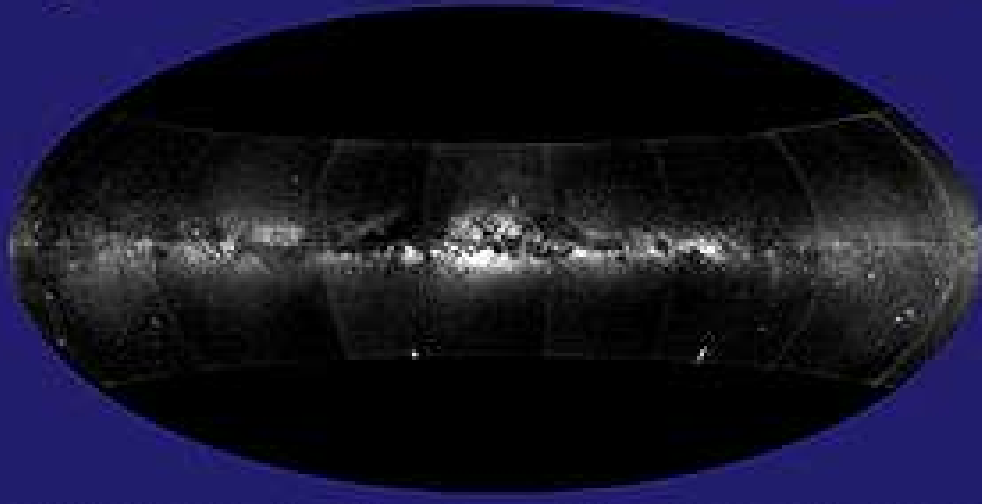


21 cm

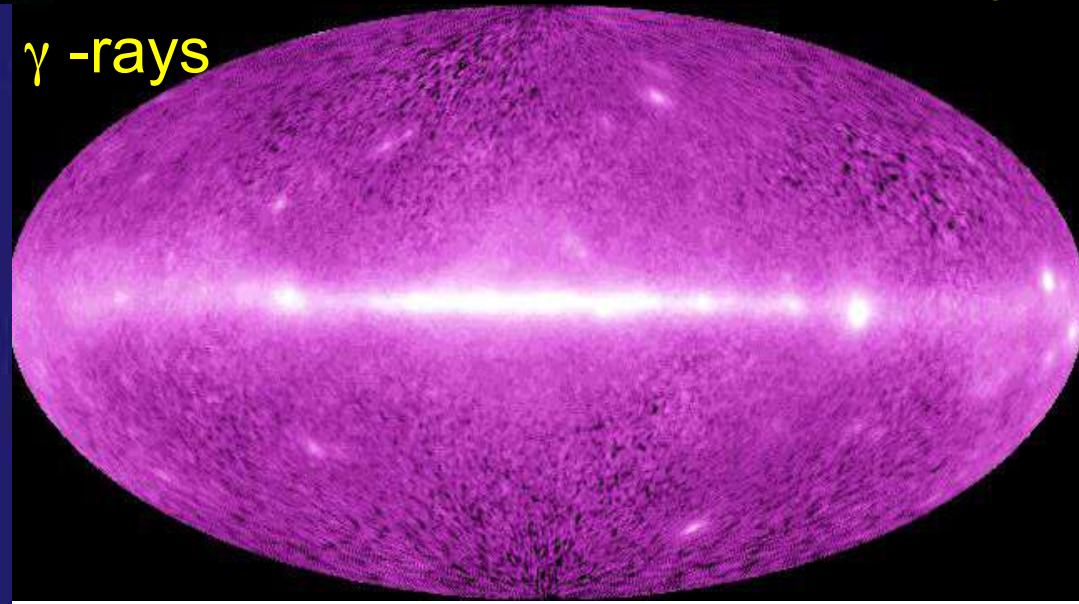


Optical

Laustsen et al. Photomosaic



γ -rays





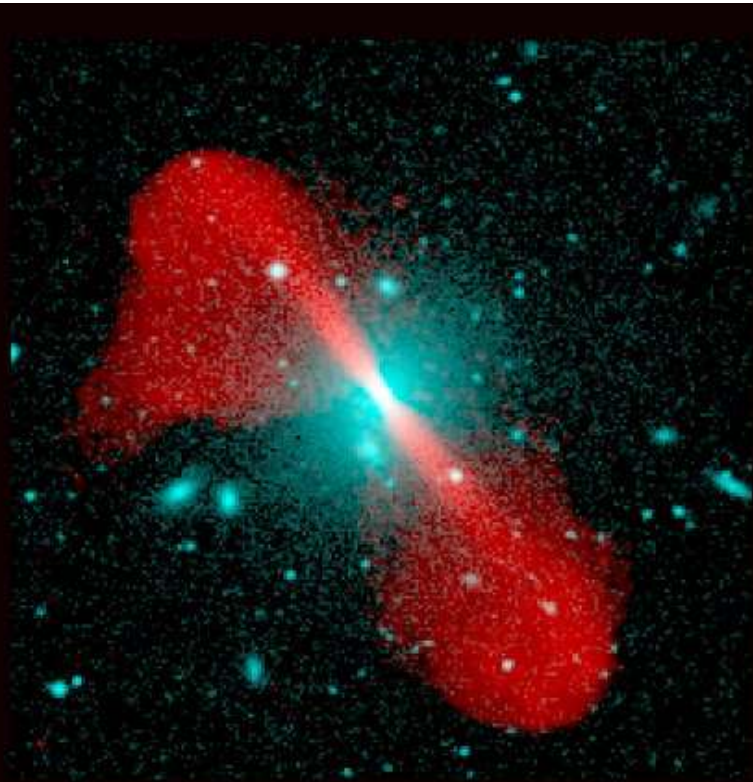
The sky is made of individual objects: sample images

Radio emission (red & orange)

.Vs.

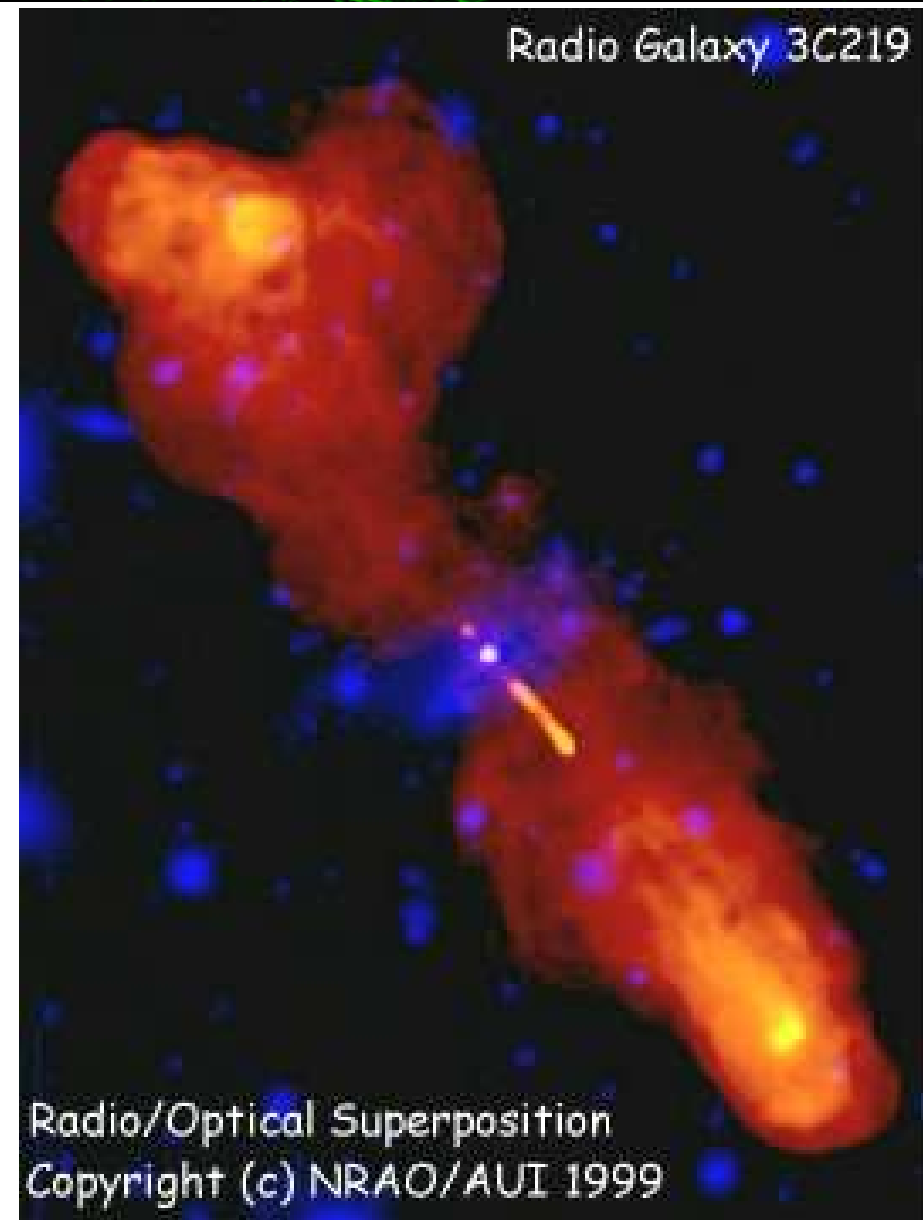
Optical emission (cyan & blue)

in *radio galaxies*



Radio Galaxy 3C296
Radio/optical superposition

Copyright (c) NRAO/AUI 1999



Radio Galaxy 3C219

Radio/Optical Superposition
Copyright (c) NRAO/AUI 1999

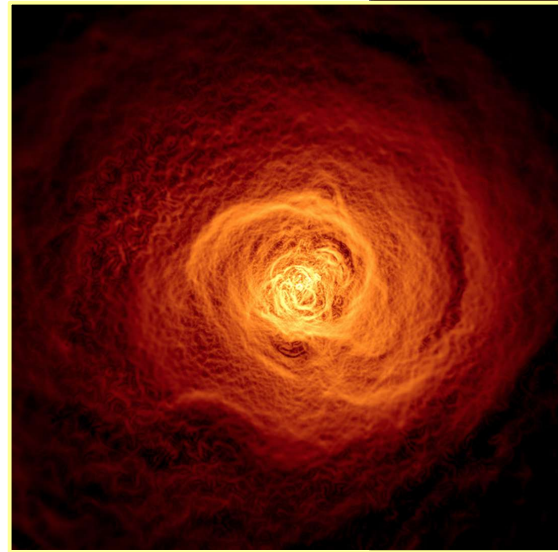
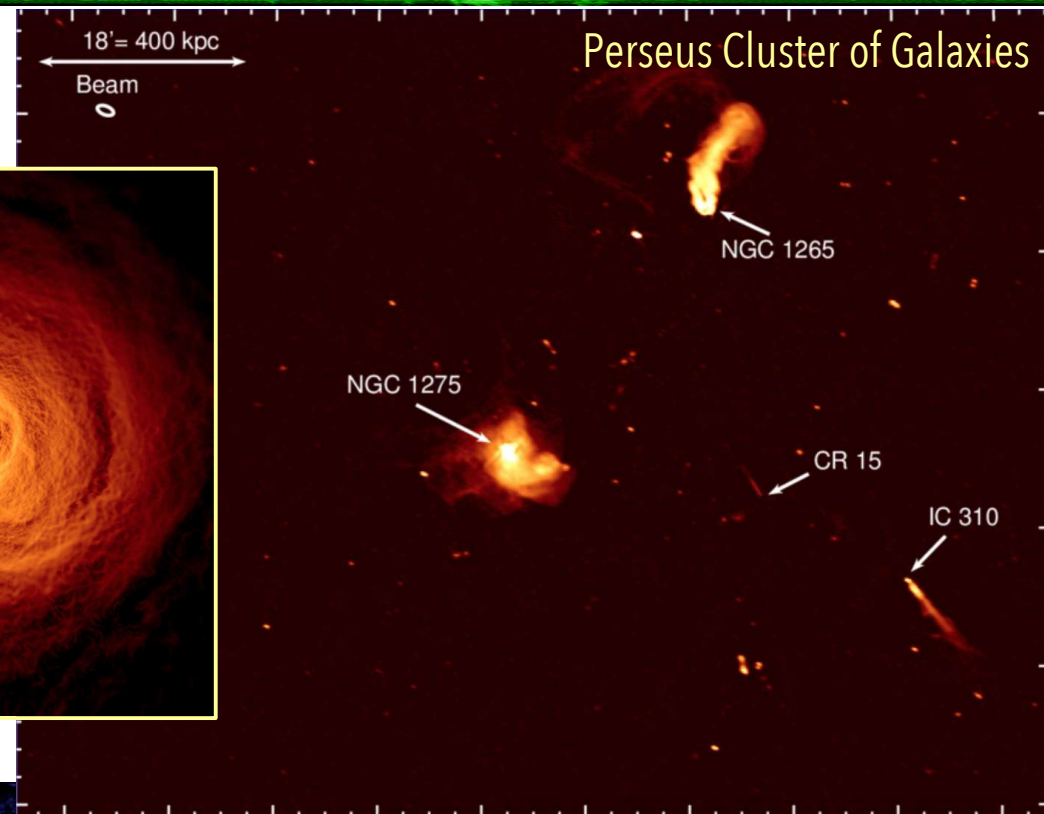


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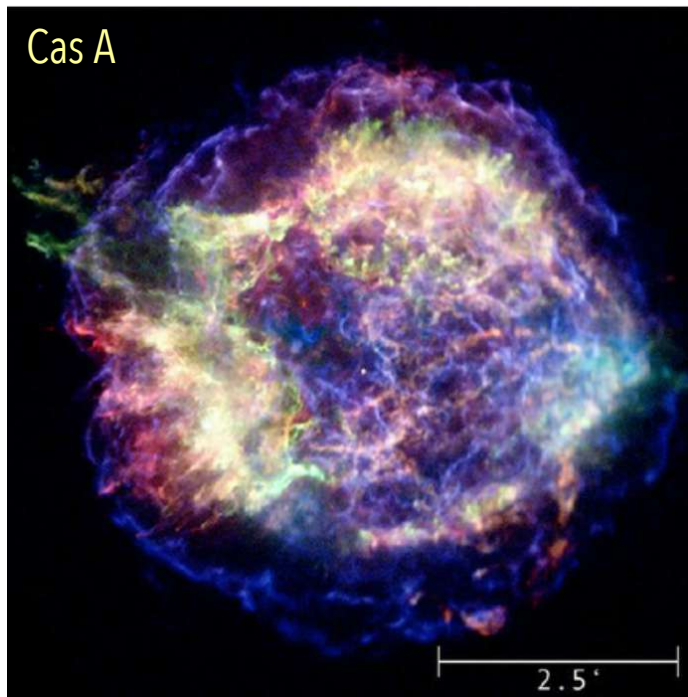
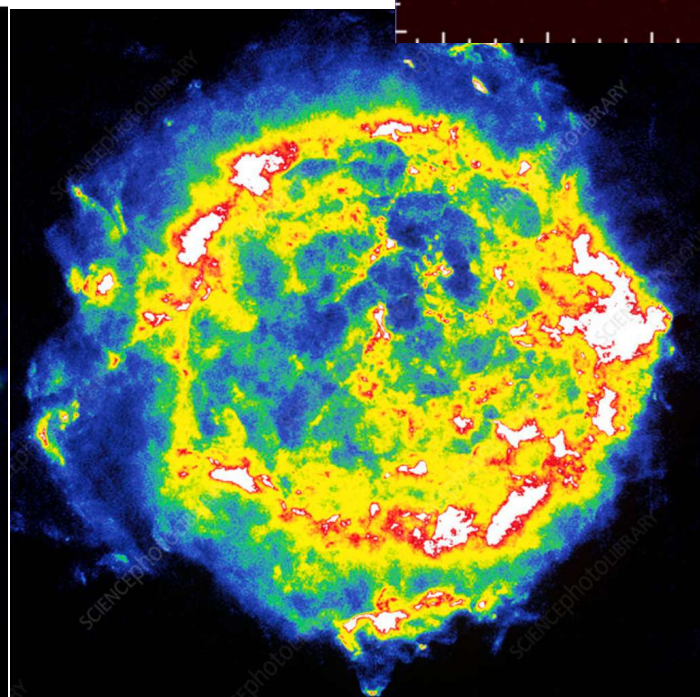
X-ray emission

1. Galaxy clusters [$\sim$ Mpc size]
(in orange) →

2. SNR (multicolor) [$\sim$ pc size]
↓



Cas A



Radio emission

↑ 1. Galaxy clusters [$\sim$ Mpc size] (in orange)

← 2. SNR (false colors) [$\sim$ pc size]



CONTENTS: **Part ONE** (~ 1.25 CFU)

MHD: Statics

HydroDynamics & neutral gas physics

MagnetoHydroDynamics & Plasma physics

(Fluid Instabilities – Just mentioned)

Part TWO (~ 5.5 CFU)

Radiative processes:

Continuous radiation from a charged particle (fundamental physics of astrophysical plasmas)

1. **Black Body**
2. **Bremsstrahlung**
3. **Synchrotron**
4. **Scattering** (diffusion processes)
5. **Cosmic Rays**
6. **Acceleration mechanism(s)** [goes with part ONE]



CONTENTS: **Part THREE** (~2.25 CFU)

The Interstellar medium & line emission/absorption

1. Plasma, Neutral medium, Dust & Molecules
2. Atomic & molecular transitions
3. Einstein's Coefficients
4. Atoms (molecules) as oscillators: permitted & forbidden lines
5. Radiative .vs. collisional (de)excitation
6. The phases of the ISM – diagnostics
7. Case studies: HI and the rotation curve in spiral galaxies
// the Stromgren sphere



Part 1: Magneto - Hydrodynamics (MHD)

Main references:

- *Clarke & Carswell – "Astrophysical Fluid Dynamics"*
- *Padmanaban – "Theoretical Astrophysics" (Vol I – Astrophysical processes) (Chap. 8 & Chap. 9)*
- *Fanti & Fanti – "Lezioni di Radioastronomia" (Chap. 8)*



(Loose) definition of an ideal fluid and caveats

Why almost all systems in astrophysics can be described with the fluid formalism

Write (simple) equations to describe the evolution of a fluid element

Determine dynamical properties of fluids (viscosity and vortices)

Determine how perturbations can propagate in fluids and the role of the "sound speed"

Supersonic perturbations and implications on the fluid physical properties

Definition of a plasma

The role of magnetic field (and why the electric field is not relevant) and its properties: diffusion, flux conservation, amplification, magnetic forces, etc.

Relevant fluid properties (based on its content and its magnetic field)

Interaction between fluids and their constituents



Basic mathematical tools required for understanding

Gradient of a scalar: $\nabla \equiv \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$ $\nabla \approx \frac{1}{L}$

the vector is perpendicular to the surface where the scalar is constant

Divergence of a vector $\vec{\nabla} \cdot \vec{h} \equiv \frac{\partial}{\partial x} h_x + \frac{\partial}{\partial y} h_y + \frac{\partial}{\partial z} h_z$ $\nabla \cdot \approx \frac{1}{L}$

Curl of a vector $\vec{\nabla} \times \vec{h} \equiv \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ h_x & h_y & h_z \end{vmatrix}$ $\vec{\nabla} \times \approx \frac{1}{L}$

∇^2 $\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \rightarrow$ $\nabla^2 \approx \frac{1}{L^2}$

Approximate relationships between operators and practical quantities:

$U = \text{velocity}; L = \text{length}; T = \text{time}$

$$\frac{\partial}{\partial x} \approx \frac{1}{L} \quad \frac{\partial}{\partial t} \approx \frac{1}{T} \approx \frac{U}{L}$$

FYI,
Never asked
@ exams



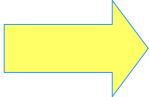
HydroDynamics (fluid mechanics)

Fluids are in form of either **liquid** or **gaseous** bodies.

Fluids are ideal macroscopic bodies with **continuous** properties (made of microscopic particles).

Hydrostatics is defined by the ideal gas equation state (P, ρ, T).

Hydrodynamics requires to know also the instantaneous velocity (P, ρ, T, v)

 $v(x, y, z, t)$ is the **velocity vector of a fluid volume element**, and not that of individual fluid particles.

Description can be based on

1. study the fluid flowing across a given place/surface [Eulerian view]
2. study a volume element in its flow/motion [Lagrangian view]

At first the magnetic field is not considered. It will be in MHD (**MagnetoHydroDynamics**)



Astrophysical fluids

Extension of the “physical” concept.

“Large” scale:

Gas/dust (ISM) in spiral (elliptical) galaxies

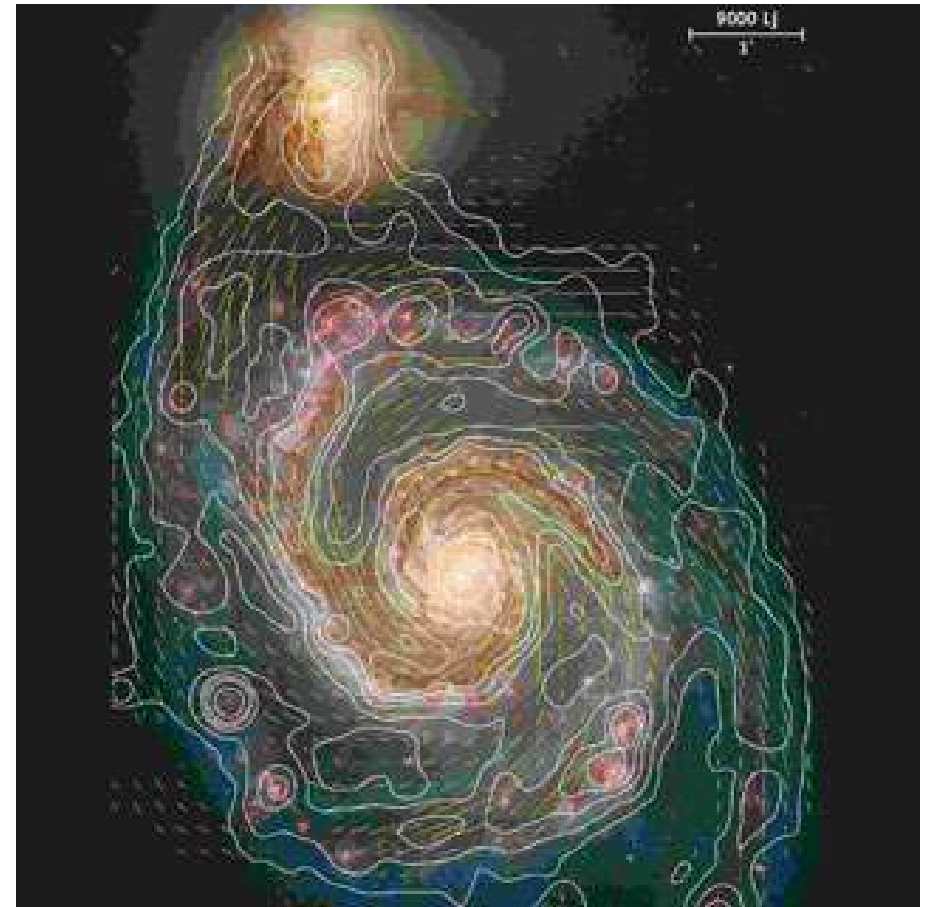
Stars in galaxies

Galaxies (& clusters) & IGM in the Universe

“Small” scale:

Stellar winds

Accretion discs & Jets, Stars, gas/dust clouds



In general, the “gaseous” fluid is more appropriate (liquid fluid describes high pressure environments, e.g. planetary & star surfaces and interiors) since the typical densities are very small. In general the fluid is inhomogeneous ($T - \rho - P$), according to its location

HIM, WIM, WNM, CNM

There are other compact bodies like WD & NS, where the equation of state is fundamental to define the fluid properties

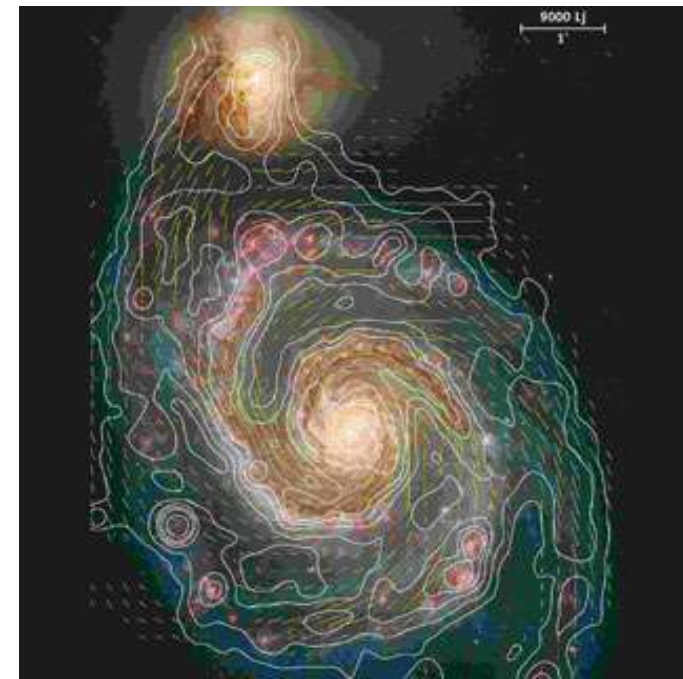
We will mainly deal with sparse fluids (low densities of “particles” wrt the total volume)

Basic concept:

Each **volume element** must be

small enough to preserve a single value for relevant quantities:

$$L_{\text{fluid}} \ll L_{\text{scale}} \sim \frac{q}{\Delta q} \quad \text{where } q = \text{generic quantity}$$



large enough to be statistically representative (large number of particles)

$$n L_{\text{fluid}}^3 \gg 1 \quad \text{where } n = \text{number density}$$

In case the fluid element has

$$L_{\text{fluid}} \gg \lambda_{\text{mfp}} \quad \text{where } \lambda_{\text{mfp}} = \text{mean free path}$$

then particles redistribute "information" and the fluid is termed collisional.

→ All this implies that a number of statistical laws hold.

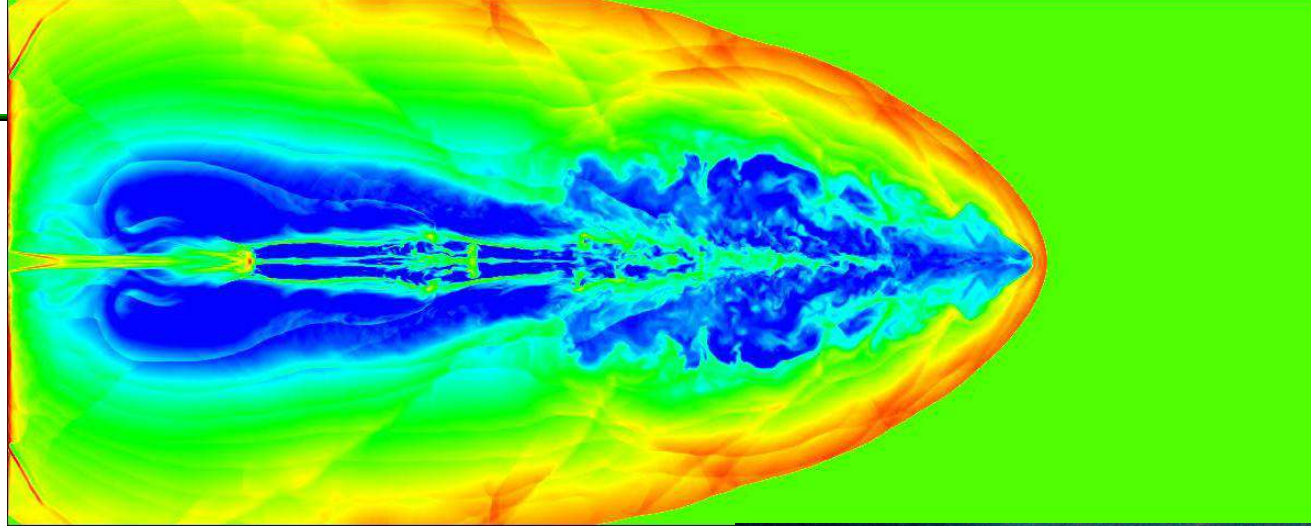
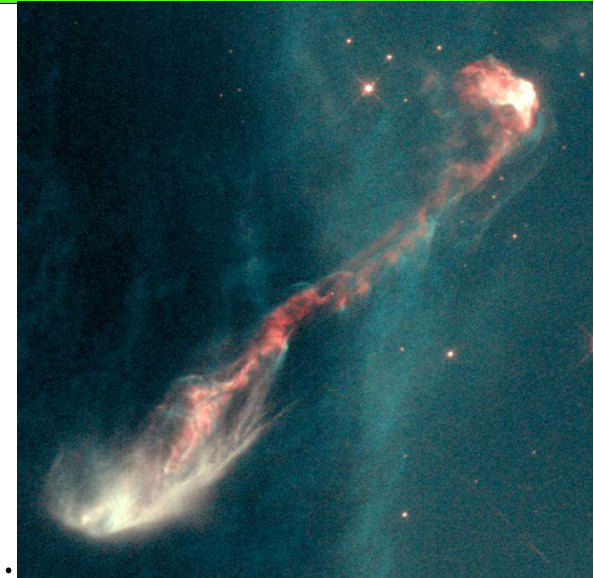
Ideal fluid

1. no internal friction

2. changes in shape
do not require work
(V remains constant)

3. [incompressible]

4. continuous, microscopic volumes (dV) still
contain large amounts of particles



➡ The vast majority of the celestial bodies (and the space between them) can be physically described with (magneto-)hydrodynamics.

Astrophysical examples:

➡ stars, stellar winds, supernovae, interstellar gas, cosmic rays, galaxies, galactic winds, intergalactic space matter (e.g. hot gas in clusters of galaxies), radio sources, etc.

➡ N.B. If a fluid is (fully) **ionized** (but overall it is neutral!), it is termed **plasma**



Fluid statics . vs. Fluid dynamics

Fluid statics studies equilibrium

local thermodynamic conditions are all you need:

state equation (e.g. $PV = nRT$) alias

$$P = f(n/V, T) = f(\rho, T)$$

Dynamics instead studies/considers motions of ''volumes''

in case of motion, velocity ($\vec{v}$) must be taken into account, in addition, and any given quantity, like pressure, P becomes

$$P = f(\rho, \vec{v}, T)$$

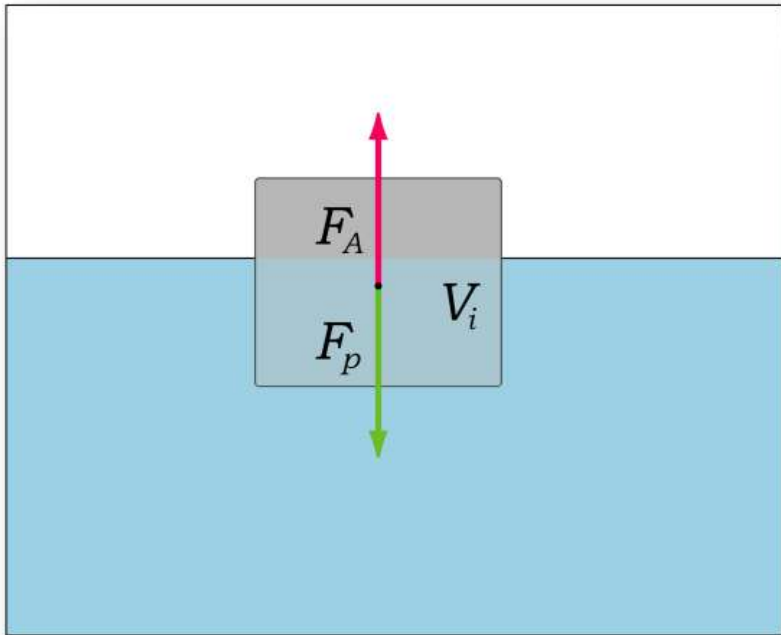


1. Pascal Principle:

In a fluid at rest, w/o any volumetric forces (e.g. gravity),

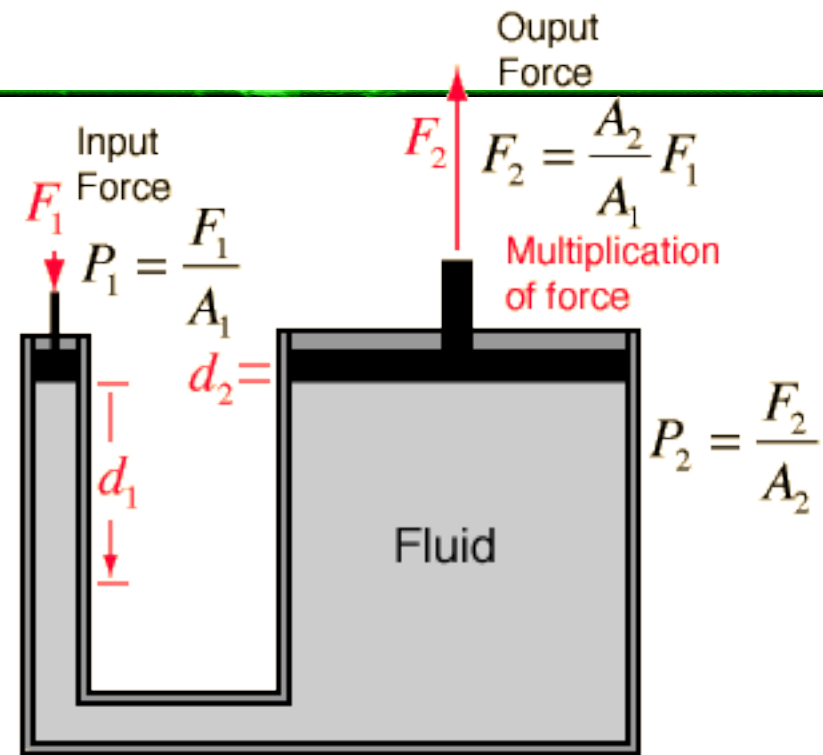
P is constant within the fluid.

A small force F_1 becomes strong in F_2



2. Archimede's Principle:

A solid body within a fluid is subject to all the forces over its surface from the fluid pressure. (A.'s push up: = weight of the liquid displaced by the body)



$$F_1 d_1 = F_2 d_2$$

$$d_1 = \frac{F_2}{F_1} d_2 = \frac{A_2}{A_1} d_2$$

You have to pay for the multiplied output force by exerting the smaller input force through a larger distance.

FYI,
not for study



3. Stevino's law:

The case of the (Earth) atmosphere):

there is a force (e.g. gravity) acting on volume,
with a component along z [i.e. $g = (0, 0, g_z)$]

$$P(z) = -g\rho z + \text{constant}$$

$$\frac{dP}{dz} = -g\rho \quad \text{where} \quad \rho = \mu N = \mu \frac{P}{kT}$$

for an isothermal atmosphere

$$\frac{dP}{P} = -\frac{\mu g}{kT} dz \quad (\mu = \text{mean molecular mass})$$

$$P = P_0 e^{-\mu g z / kT} \quad \rho = \rho_0 e^{-\mu g z / kT}$$

$$P_0 = 1.033 \text{ dyne} \cdot \text{cm}^{-2} = 1.013 \cdot 10^5 \text{ N} \cdot \text{m}^{-2} (\text{Pa})$$

$$\rho_0 = 1.29 \cdot 10^{-3} \text{ g} \cdot \text{cm}^{-3} = 1.29 \text{ kg} \cdot \text{m}^{-3}$$

Components in Dry Air	Volume Ratio compared to Dry Air	Molecular Mass - M (kg/kmol)	Molecular Mass in Air
Oxygen	0.2095	32.00	6.704
Nitrogen	0.7809	28.02	21.88
Carbon Dioxide	0.0003	44.01	0.013
Hydrogen	0.0000005	2.02	0
Argon	0.00933	39.94	0.373
Neon	0.000018	20.18	0
Helium	0.000005	4.00	0
Krypton	0.000001	83.8	0
Xenon	$0.09 \cdot 10^{-6}$	131.29	0
Total Molecular Mass of Air			28.97



FYI,
not for study

The amount of mass within a given volume V_0 is: $\int_{V_0} \rho dV$

The mass flowing through a surface $d\vec{S}$ per unit time is:
 $\rho \vec{v} \cdot d\vec{S} = \rho (\vec{v} \cdot \hat{n}) dS$ [If > 0 outflowing w.r.t. V_0]

Total mass outflowing V_0 :

$$\oint \rho \vec{v} \cdot d\vec{S} = \int_{V_0} \vec{\nabla} \cdot (\rho \vec{v}) dV \quad (1) \quad (\text{Gauss' theorem})$$

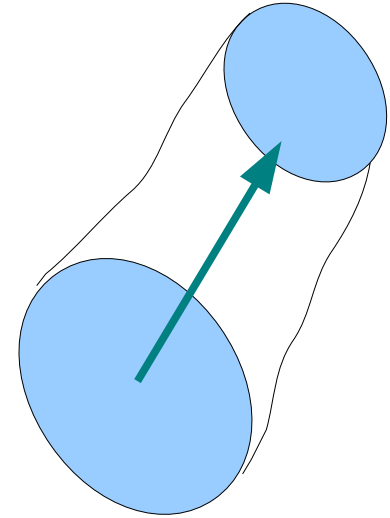
Decrease of the mass fluid in V_0 per unit time:

$$-\frac{\partial}{\partial t} \int_{V_0} \rho dV = - \int_{V_0} \frac{\partial \rho}{\partial t} dV \quad (2)$$

Equating (1) and (2)

$$- \int_{V_0} \frac{\partial \rho}{\partial t} dV = \int_{V_0} \vec{\nabla} \cdot (\rho \vec{v}) dV$$

$$\int_{V_0} \left[\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) \right] dV = 0$$



Must hold for any volume $\rightarrow$ integrand must vanish:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}(\rho \vec{v}) = 0$$

$$\frac{\partial \rho}{\partial t} + \rho \vec{\nabla} \vec{v} + \vec{v} \cdot \vec{\nabla} \rho = 0 \quad \text{Eulerian}$$

the quantity $\rho \vec{v} = \vec{j}$ is also known as **mass flux density**

Let's introduce the **convective derivative** ** $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla}$

the **equation of continuity** can also be written as

$$\frac{D\rho}{Dt} + \rho \vec{\nabla} \vec{v} = 0 \quad \text{Lagrangian}$$

** it is like we concentrate our attention to a given mass (volume) element following its motion. Such derivative represents the variation of the density as co-moving with it.

application (1):

during Δt the fluid moves of a certain amount

$$\Delta x_1 = v_1 \Delta t \text{ in } S_1 \text{ and of } \Delta x_2 = v_2 \Delta t \text{ in } S_2$$

with a mass motion of

$$\Delta m_1 = \rho_1 S_1 \Delta x_1 = \rho_1 S_1 v_1 \Delta t$$

$$\Delta m_2 = \rho_2 S_2 \Delta x_2 = \rho_2 S_2 v_2 \Delta t$$

in case of mass conservation $\Delta m_1 = \Delta m_2$ namely

$$\rho_1 S_1 v_1 = \rho_2 S_2 v_2$$

i.e. $\rho S v = \text{constant}$

In case of incompressible fluids ($\rho = \text{constant}$)

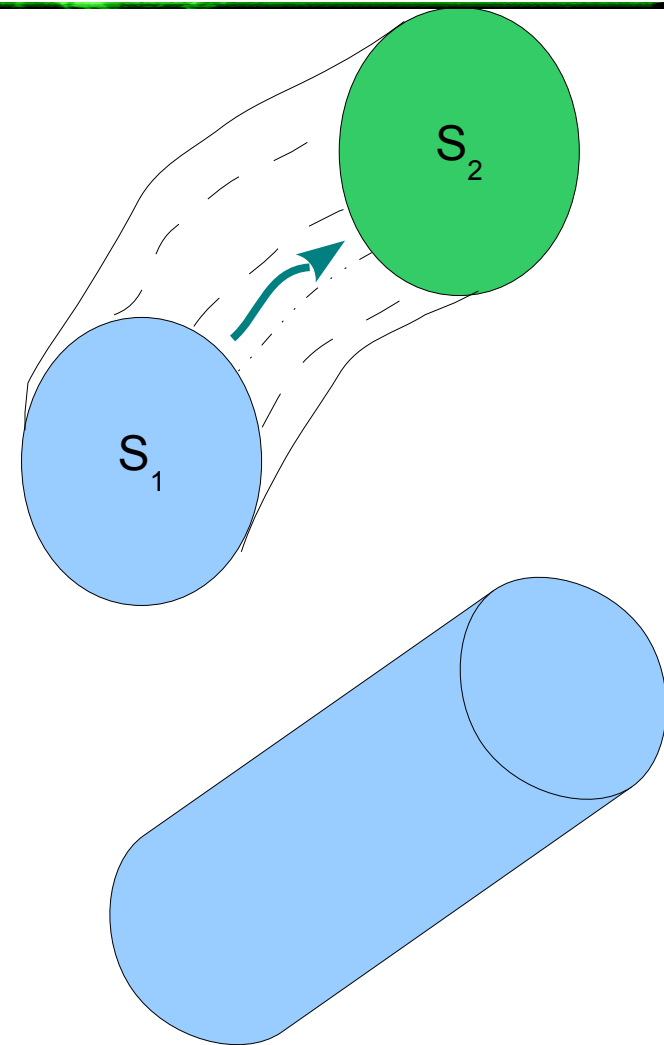
$$S v = \text{constant} = Q$$

in case of variable cross section of the pipe:

$$v \propto \frac{1}{S}$$

i.e. velocity increases when the cross section narrows

Astrophysical example: particles in a jet are accelerated in case the cross section gets smaller and smaller





application (2) : gravity plays a role!

From eq.of.c. $dm_1 = dm_2 = dm$

if $\rho = \text{constant} \rightarrow S_1 v_1 = S_2 v_2$

there is a variation of kinetic energy

$$\Delta K = \frac{1}{2} dm (v_2^2 - v_1^2) = L_p + L_g$$

$$L_g = dm g(z_1 - z_2)$$

$$L_p = p_1 S_1 v_1 dt - p_2 S_2 v_2 dt = p_1 V - p_2 V$$

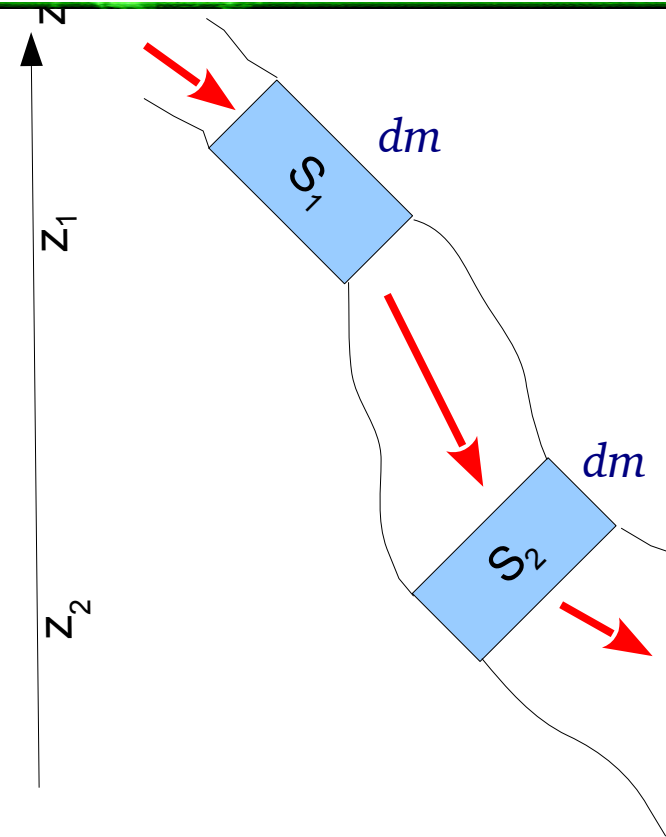
$$\frac{1}{2} dm (v_2^2 - v_1^2) = dm g(z_1 - z_2) + (p_1 - p_2) V \quad \text{but} \quad V \approx \frac{dm}{\rho}$$

$$\frac{1}{2} \frac{v_1^2}{g} + \frac{p_1}{\rho g} + z_1 = \frac{1}{2} \frac{v_2^2}{g} + \frac{p_2}{\rho g} + z_2 \quad [\text{Bernoulli's equation}]$$

stop
height

piezometric
height

height



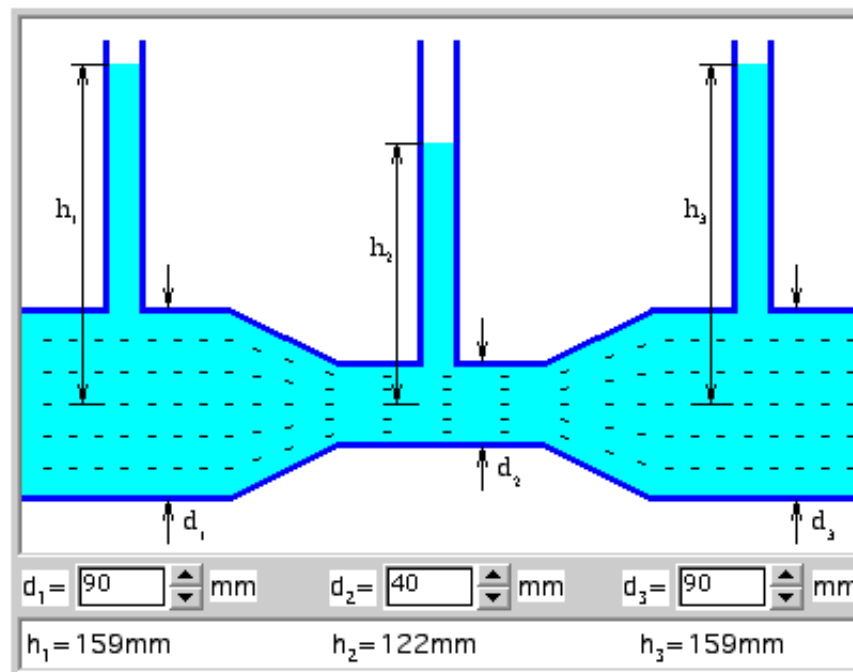
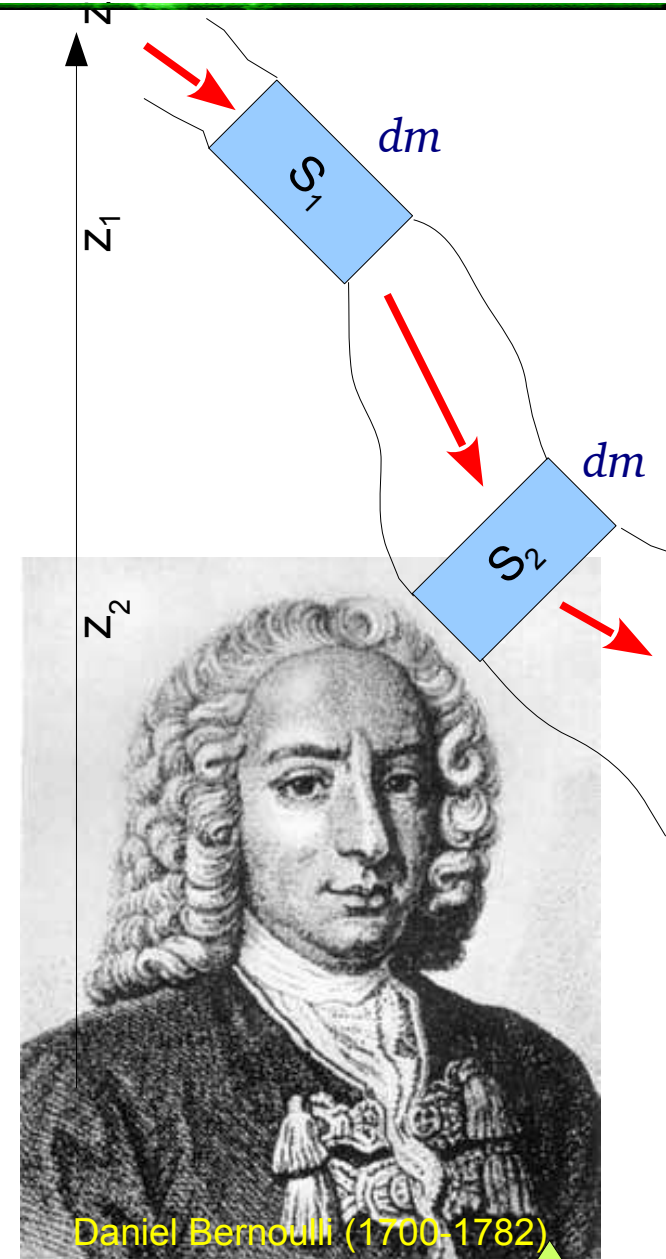
FYI,
not for study

it follows that:

$$\frac{1}{2} \frac{v^2}{g} + \frac{p}{\rho g} + z = \text{const}$$

$$\frac{1}{2} \rho v^2 + p + \rho g z = \text{const}$$

(other forms of Bernoulli's equation)



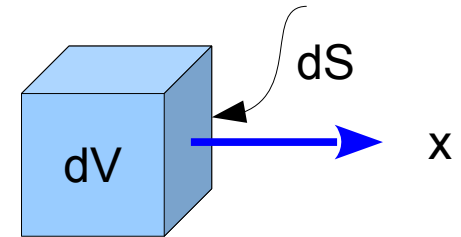
Daniel Bernoulli (1700-1782)

FYI,
not for study

Euler's equation (momentum conservation) (equivalent to $F=ma$)

1-D: along x , considering all the acting forces:

$$(1) \quad dF_x = dm \frac{dv_x}{dt} = \rho dV \frac{dv_x}{dt}$$



and we must consider the forces f_x acting on the volume dV (i.e. gravity)
and on the surface dS (i.e. pressure)

$$(2) \quad dF_x = f_x dV + [p(x) - p(x+dx)]dS =$$

$$= f_x dV + \frac{[p(x) - p(x+dx)]}{dx} dS dx = f_x dV - \frac{dp}{dx} dV = \left(f_x - \frac{dp}{dx} \right) dV$$

From (1) and (2)
$$\rho dV \frac{dv_x}{dt} = \left(f_x - \frac{dp}{dx} \right) dV \quad \rightarrow \quad \frac{dv_x}{dt} = \left(\frac{1}{\rho} f_x - \frac{1}{\rho} \frac{dp}{dx} \right)$$

3-D: The same also along y and z .

Then, using the vector formalism:

$$\frac{d\vec{v}}{dt} = \left(\frac{1}{\rho} \vec{f} - \frac{1}{\rho} \vec{\nabla} p \right)$$

The Lagrangian view follows the fluid element during the motion

The Eulerian view studies the velocity field

$$\frac{d\vec{v}}{dt} = \left[\frac{1}{\rho} \vec{f} \right] - \frac{1}{\rho} \vec{\nabla} p \quad (3)$$

refers to the rate of change of a given fluid volume dV as it moves about in space (which is the goal) and not the rate of change of fluid velocity at a fixed point in space.

$d\vec{v}$ over time interval dt is composed of two parts:

1. change (during dt) at a fixed point in space

2. difference between velocities (at the same instant) at two points dr apart where dr is the distance covered during dt

$$1. \quad \frac{\partial \vec{v}}{\partial t} \Big|_{x,y,z=\text{constant}} dt$$

$$2. \quad dx \frac{\partial \vec{v}}{\partial x} + dy \frac{\partial \vec{v}}{\partial y} + dz \frac{\partial \vec{v}}{\partial z} = (d\vec{r} \cdot \vec{\nabla}) \vec{v}$$

$$d\vec{v} = \frac{\partial \vec{v}}{\partial t} dt + (d\vec{r} \cdot \vec{\nabla}) \vec{v}$$

let's divide by dt

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v}$$

Then substituting in (3) we get the Euler's equation (1755), also known as momentum conservation equation:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \left[\frac{1}{\rho} \vec{f} \right] - \frac{1}{\rho} \vec{\nabla} p$$

$$\frac{D\vec{v}}{Dt} = \left[\frac{1}{\rho} \vec{f} \right] - \frac{1}{\rho} \vec{\nabla} p$$

includes all the external (non -HD) forces.

In a gravitational field alone

$$\vec{f} = -\rho \vec{\nabla} \phi$$



In case also **friction** (depending on v^2) and other non-conservative forces are considered, new coefficients η and η' are introduced:

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{D \vec{v}}{Dt} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\eta}{\rho} \nabla^2 \vec{v} + \frac{\eta'}{\rho} [\dots \vec{H} \dots]$$

if we forget magnetic forces

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{D \vec{v}}{Dt} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\eta}{\rho} \nabla^2 \vec{v}$$

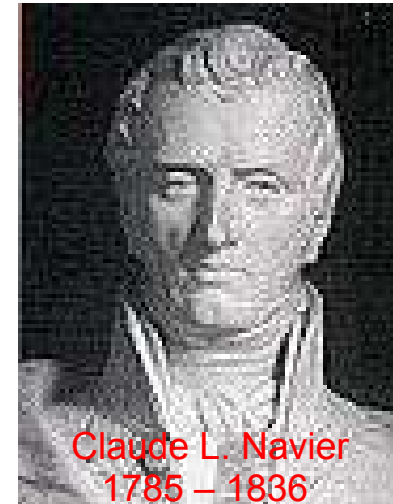
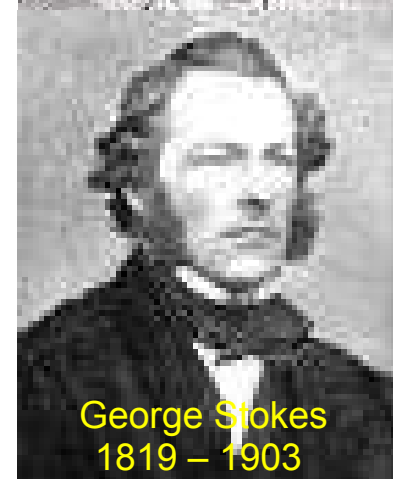
which is known as **Navier-Stokes equation**

$\nu = \eta / \rho$ is termed **kinematic viscosity**.

the quantity $R = \frac{\rho (\vec{v} \cdot \vec{\nabla}) \vec{v}}{\eta \nabla^2 \vec{v} + \eta' \dots}$ is the **Reynolds' number**

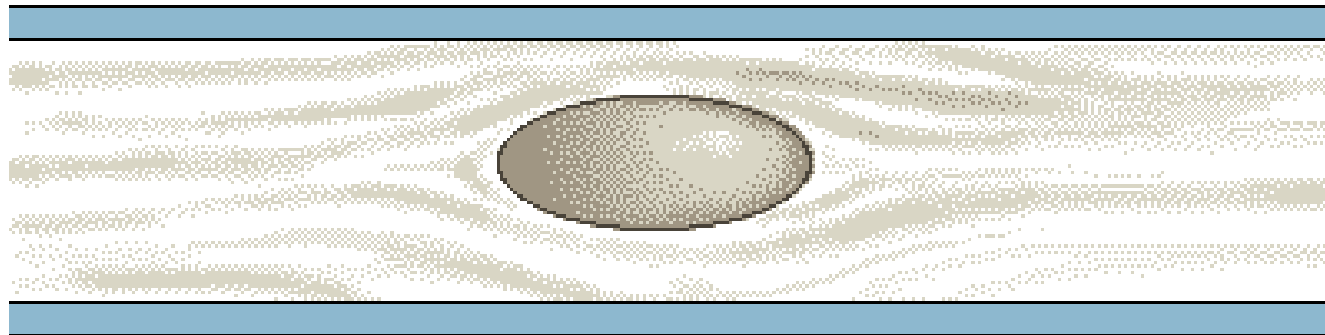
It represents the balance between inertia and viscosity and then between **turbulence** (low viscosities, high R) and **laminar motion** (high η , low R).

we will come back to this again ...

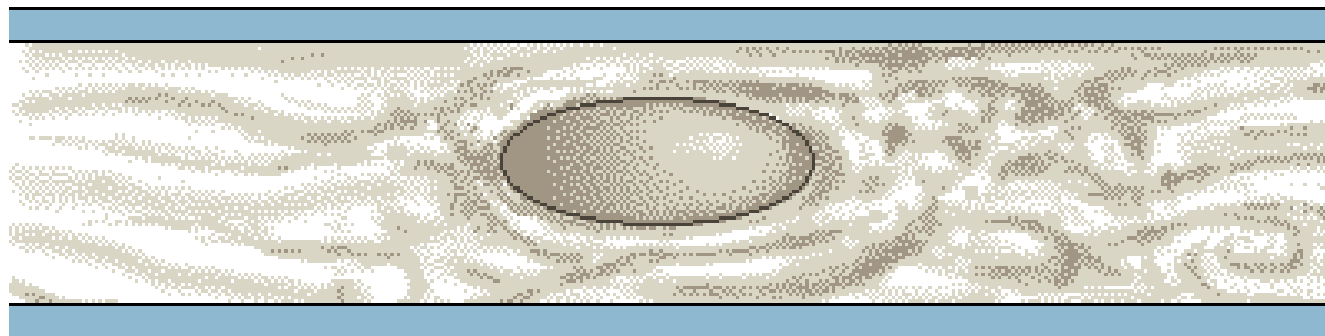


$$R = \frac{\rho (\vec{v} \cdot \vec{\nabla}) \vec{v}}{\eta \nabla^2 \vec{v} + \eta' \dots}$$

Reynolds' number



Low velocity

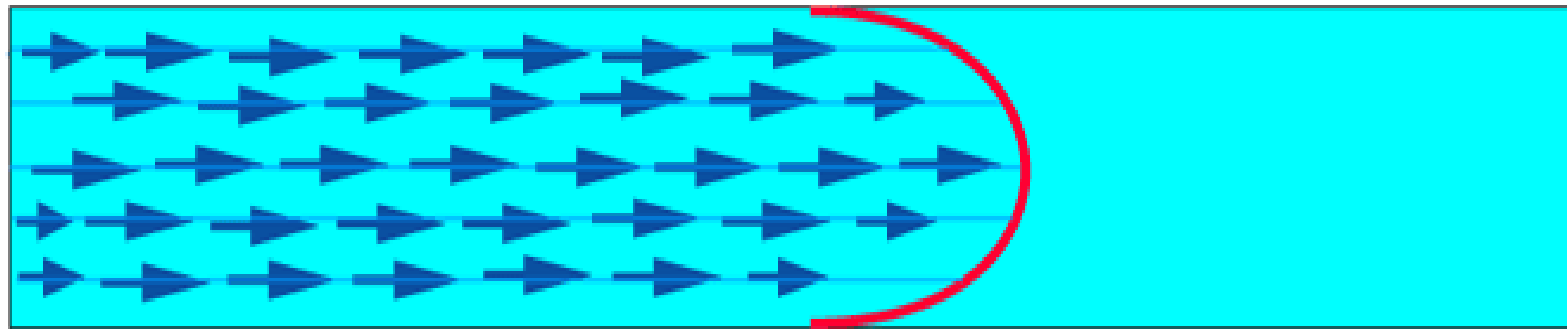


High velocity with turbulence

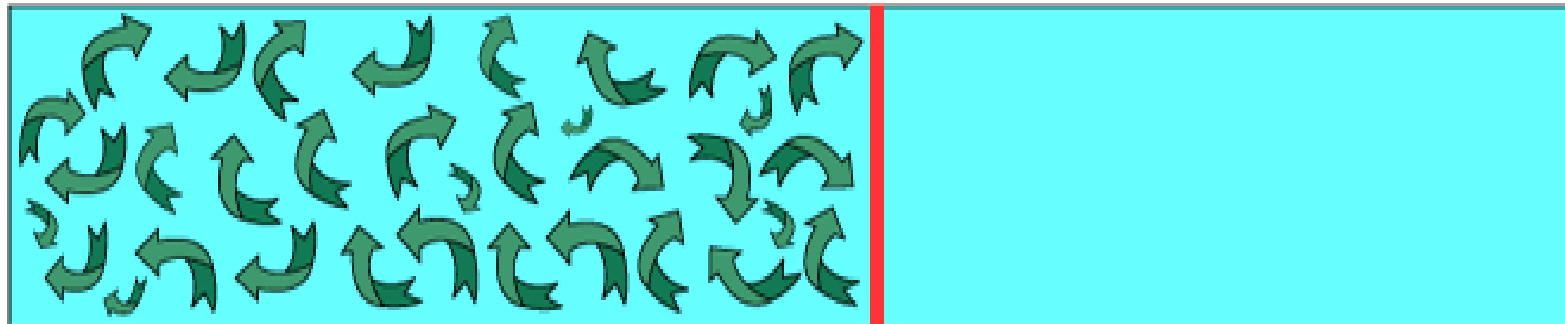
Laminar Flow



freshgasflow.com



Turbulent Flow



Think to a river:

- In case of draught, the flow is laminar (slow velocity of the flowing water)
- In case of (very) abundant rain the flow is turbulent (high velocity, and friction creates vortices)

For a given surface, the fluid contributes with an isotropic pressure (thermodynamics!) arising from the **random** (thermal) motions of the particles (P_{fluid})

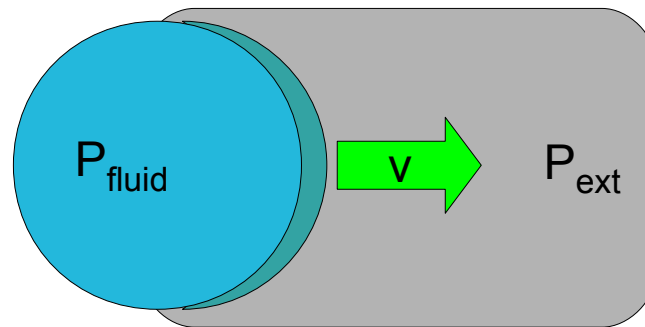
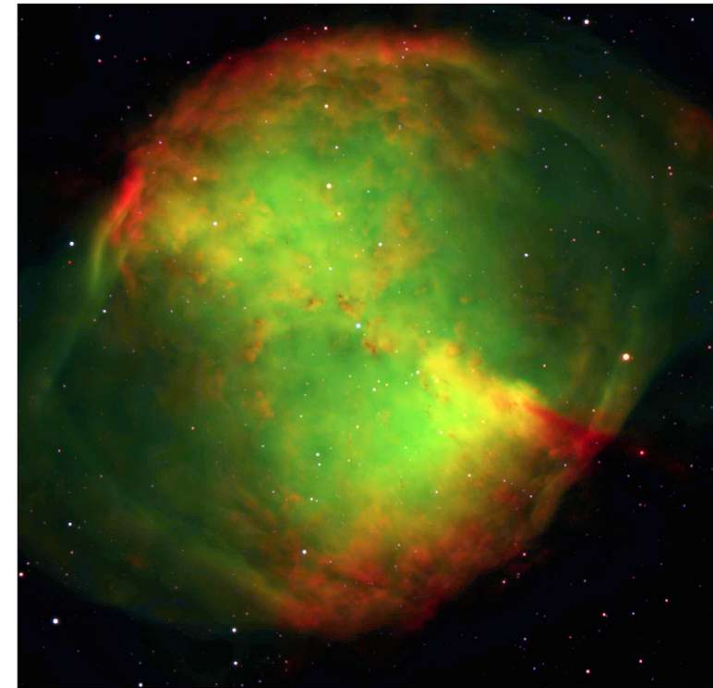
In case the fluid is in motion, at a speed v , there is an additional term ρv^2 , arising from the **bulk** motion of the fluid.

This is known as **ram pressure**, and rules the motion of a fluid within another fluid.

The surface separating two fluids reaches the pressure equilibrium:

$$P_{\text{fluid}} \Leftrightarrow P_{\text{ext}} + \rho v^2$$

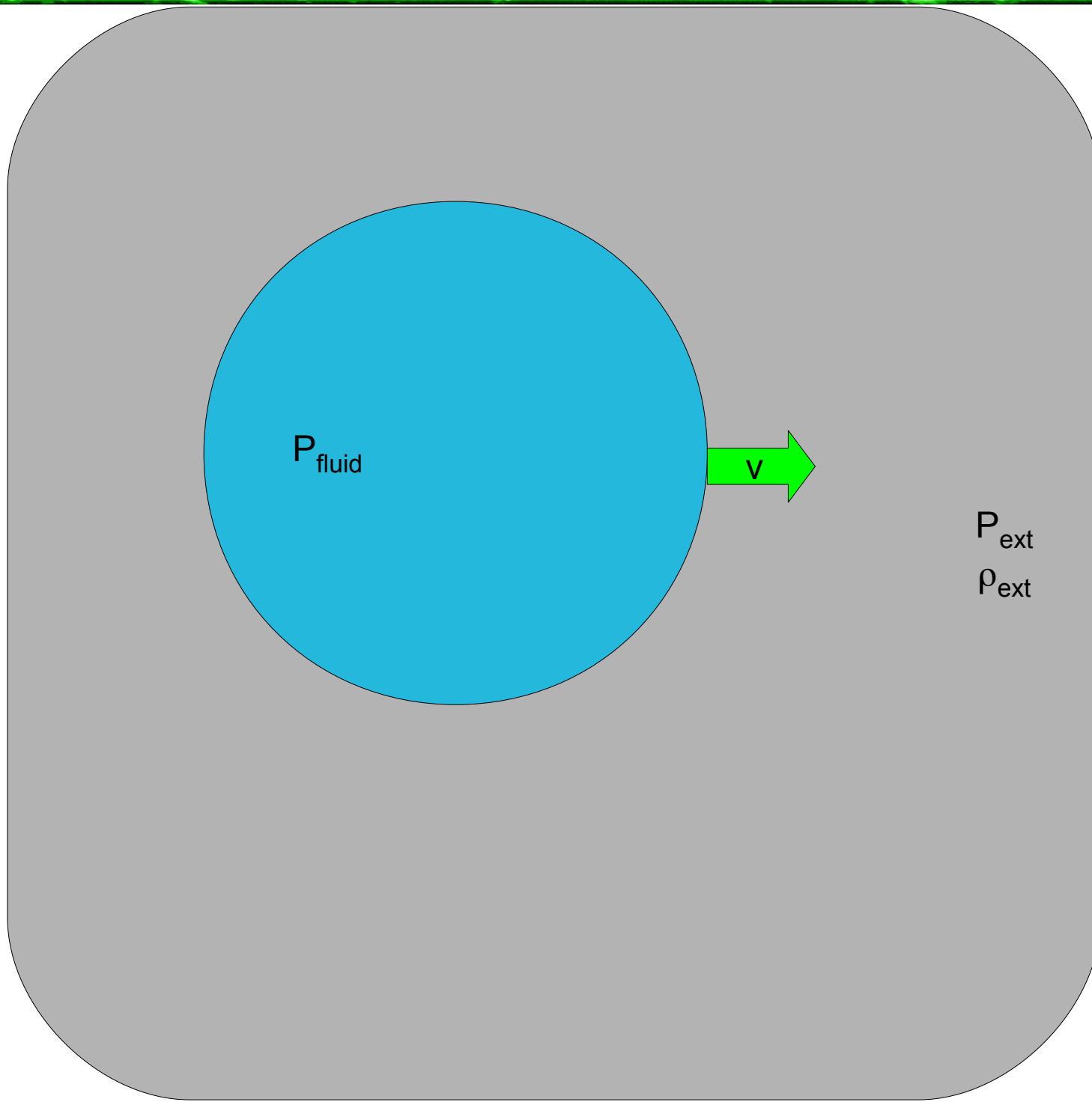
There are several examples of ram pressure in astrophysics.



Planetary Nebula NGC 6853 (M 27) - VLT UT1+FORIS1

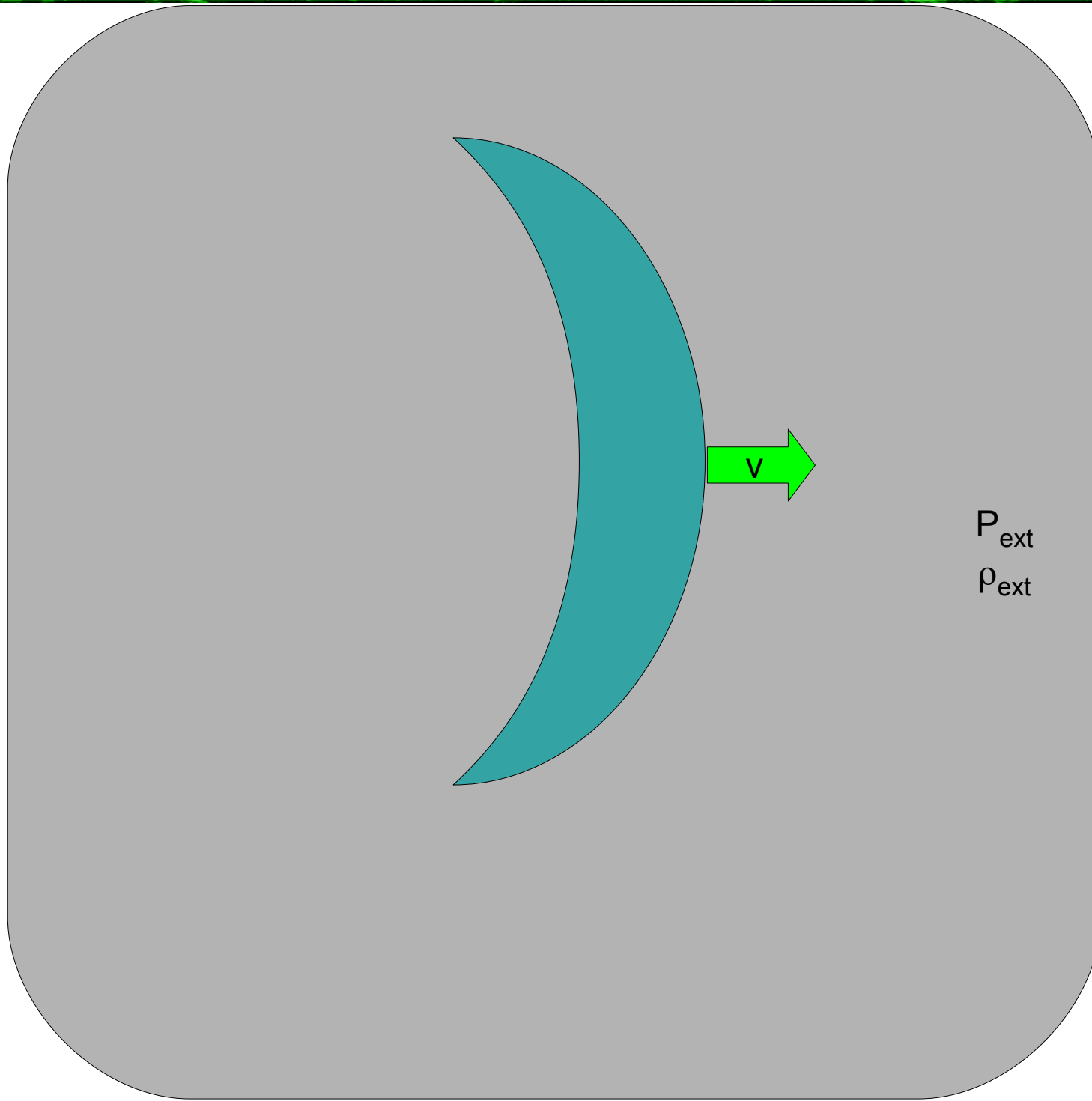


Ex (1): Which is the evolution of a bubble of a given fluid if it starts to move within another (external) fluid?



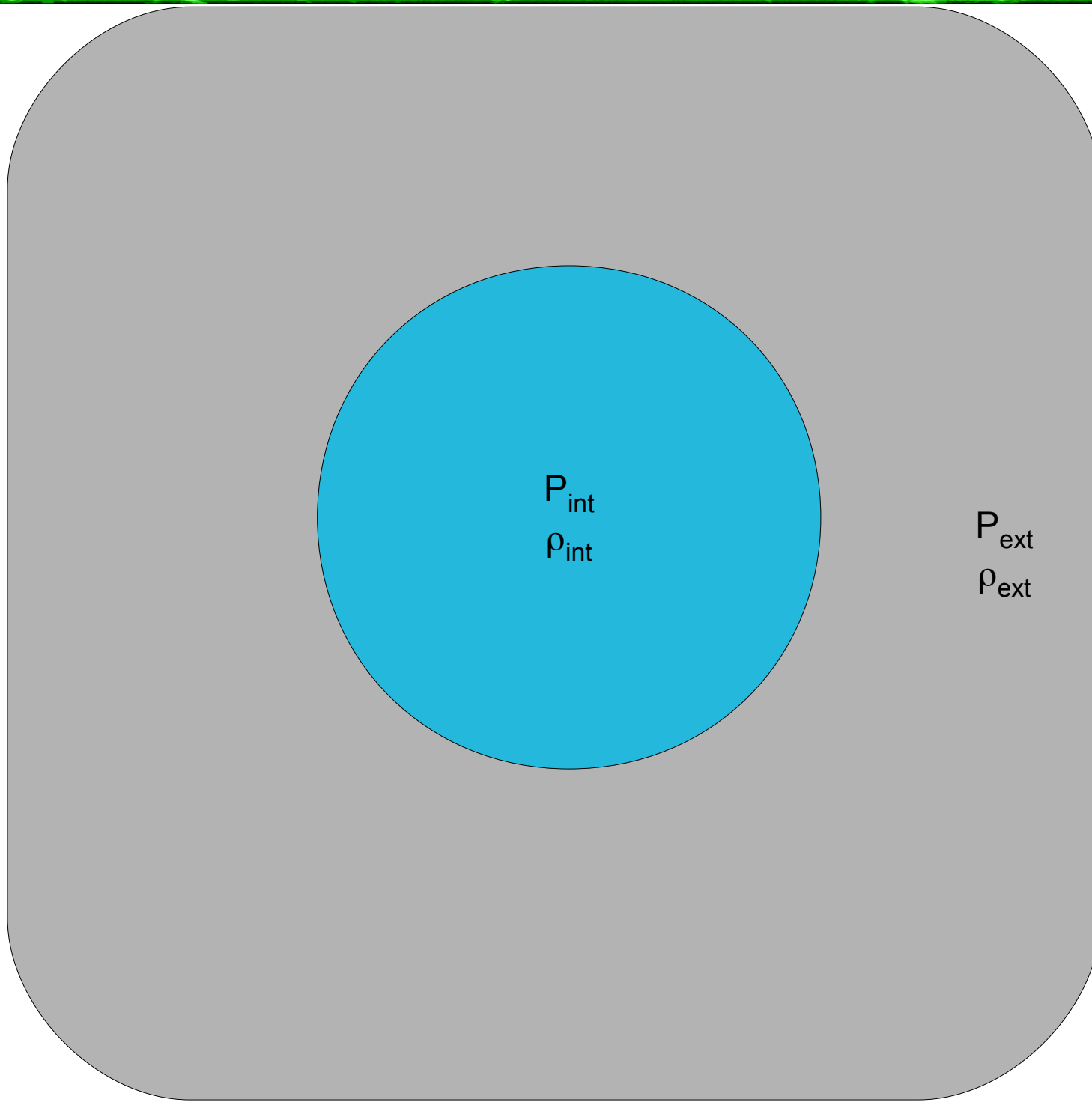


Velocity is a vector!





Ex (2): Which is the evolution of a (still) bubble of a given fluid expanding (shrinking) within another (external) fluid?



It is the application of the 1st law of thermodynamics to fluid mechanics:

u = internal energy per unit volume

$h = p + u$ = enthalpy per unit volume ($H = pV + U$)

The conservation of energy can be written as

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + u \right) = - \vec{\nabla} \cdot \left[\vec{v} \left(\frac{\rho v^2}{2} + h \right) \right] = - \vec{\nabla} \cdot \left[\vec{v} \left(\frac{\rho v^2}{2} + u + p \right) \right]$$

Energy flux vector

To understand the meaning of this equation let's integrate over a given volume V

$$\frac{\partial}{\partial t} \int \left(\frac{1}{2} \rho v^2 + u \right) dV = - \int \vec{\nabla} \cdot \left[\vec{v} \left(\frac{\rho v^2}{2} + h \right) \right] dV$$

$$\frac{\partial}{\partial t} \int \left(\frac{1}{2} \rho v^2 + u \right) dV = - \underbrace{\oint \left[\vec{v} \left(\frac{\rho v^2}{2} + h \right) \right] dS}_{\text{amount of energy flowing out the volume } V \text{ per unit time}}$$

amount of energy flowing out
the volume V per unit time

FYI,
Never asked
@ exams

Further, we consider ϵ and w the **energy** and the **enthalpy per unit mass** (instead of the values u and h per unit volume)

$$\frac{\partial}{\partial t} \left(\frac{1}{2} \rho v^2 + \rho \epsilon \right) = - \nabla \cdot \left[\rho \vec{v} \left(\frac{v^2}{2} + w \right) \right]$$

conversely, if $V^* \approx 1/\rho$ is the specific volume and P^* the external pressure (modulus) and Q^* the heat source $\approx$ (Principle I of TD)

$$\frac{D\epsilon}{Dt} + P^* \frac{DV^*}{Dt} = Q^*$$

further expressions include also entropy.

Other expressions of the energy equation (also in non-conservative form) may be given: additional terms must be added on the right side of the equation

1. Equation of continuity
$$\frac{\partial \rho}{\partial t} + \rho \vec{\nabla} \cdot \vec{v} + \vec{v} \cdot \vec{\nabla} \rho = 0$$

It is a scalar equation and contains ρ and $\vec{v}$, namely 4 variables

2. Euler's (Navier-Stokes') equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = \frac{D \vec{v}}{Dt} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\eta}{\rho} \nabla^2 \vec{v}$$

It is a vector relation and adds the variable p

3. Equation of energy
$$\frac{D \epsilon}{Dt} + p^* \frac{DV^*}{Dt} = Q^*$$

It is a scalar equation and introduces one new variable (h or u)

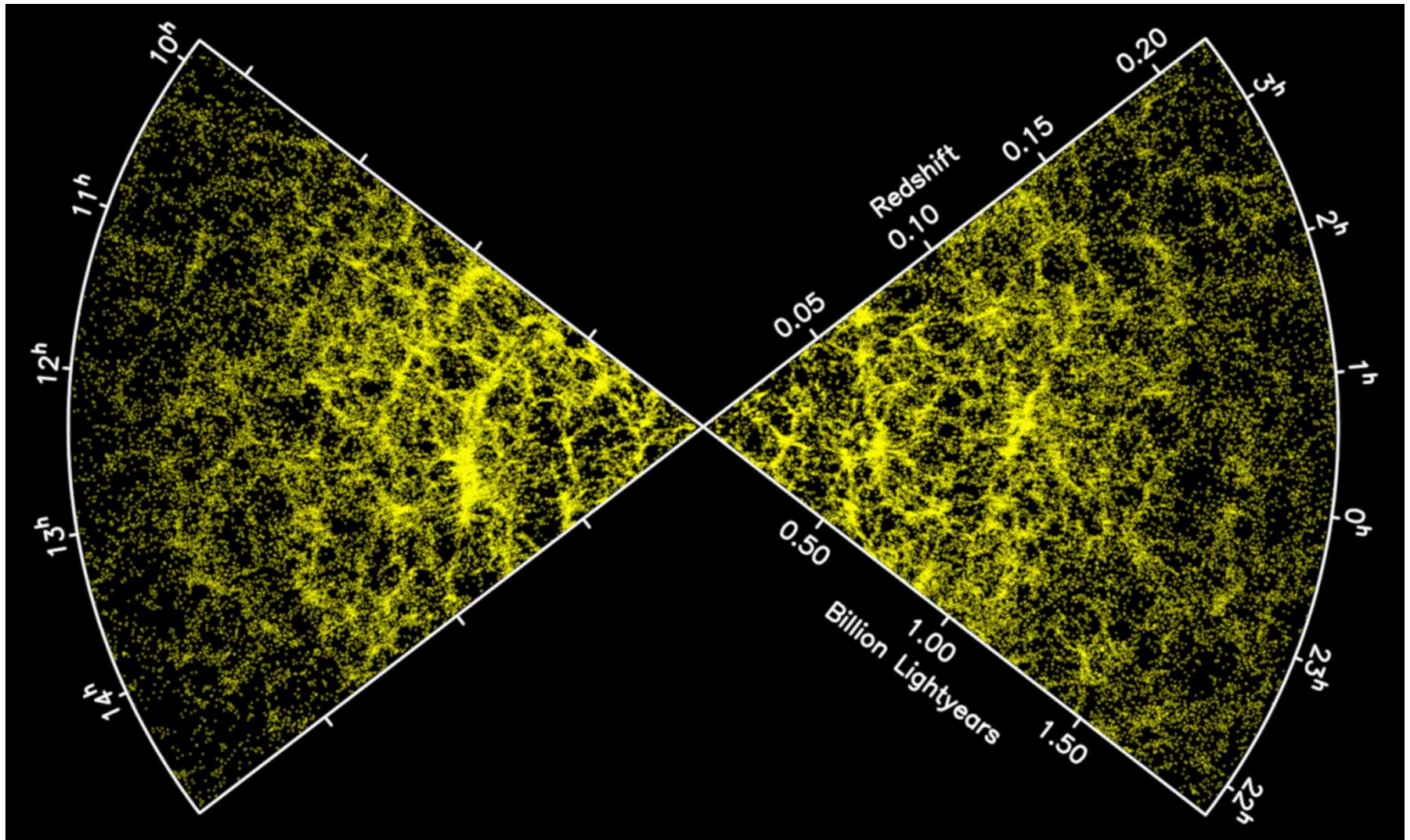
Therefore we have 5 independent equations with 6 variables and one more relation is required to solve the full set.

In general the **fluid state equation** is required to fix this problem.

The 2dF Galaxy Redshift Survey

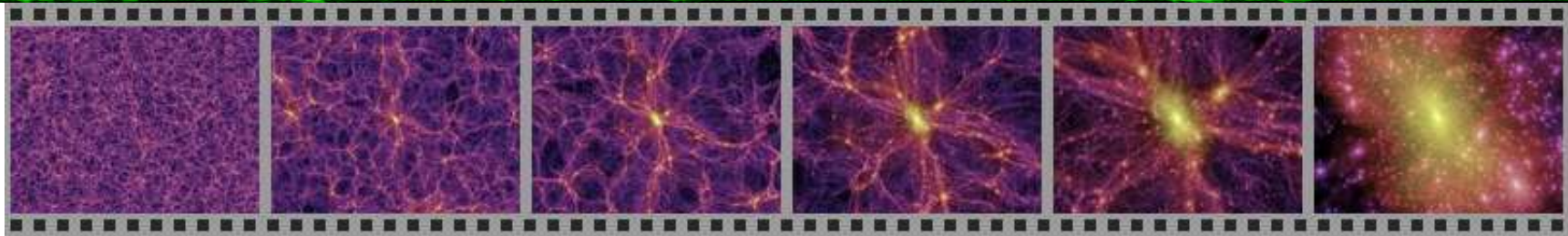
<http://www.roe.ac.uk/~jap/2df/>

How **observations** unveil the Large Scale Structure (galaxies only), the **local Universe**



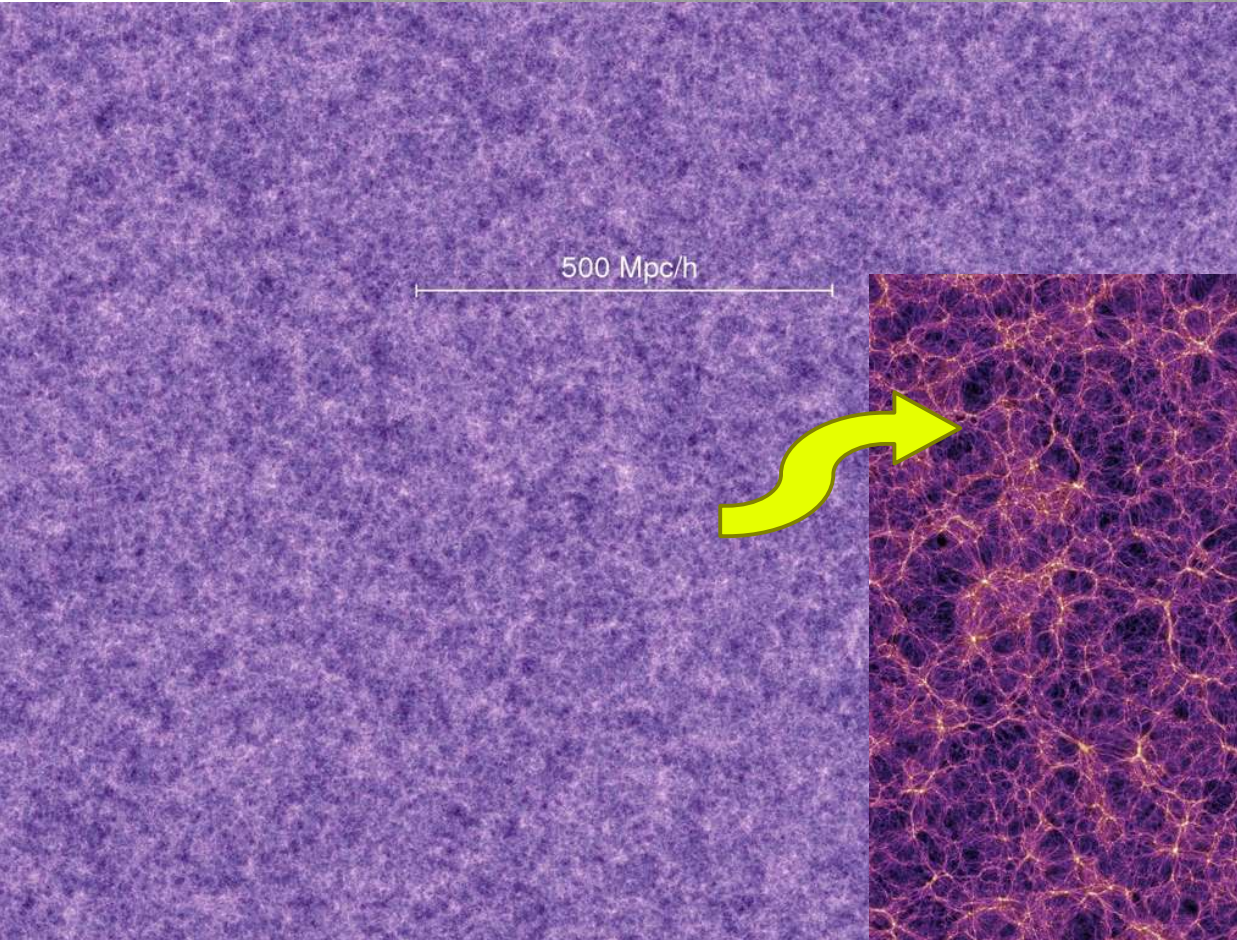
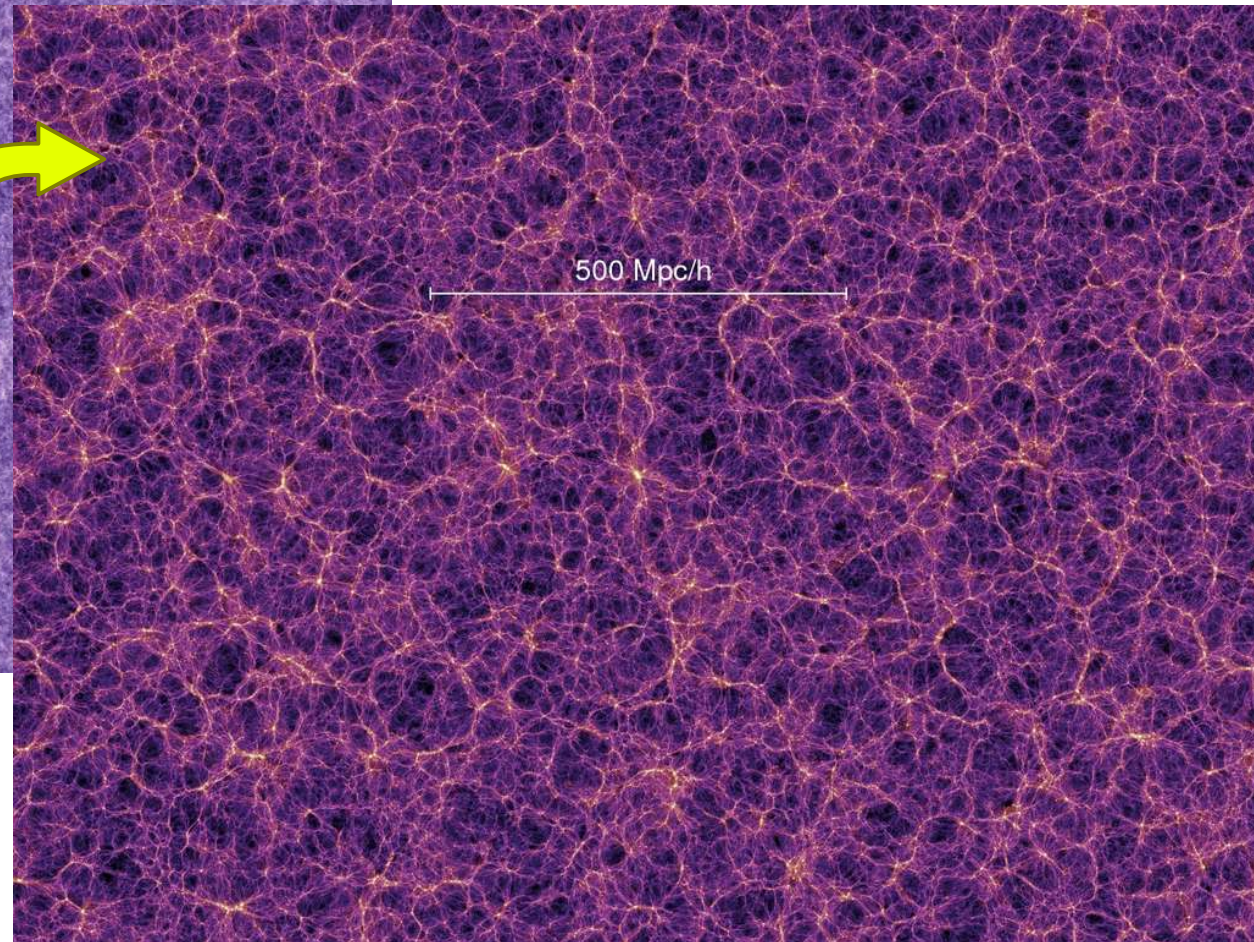
The 2dF redshift survey used the two-degree field spectroscopic facility on the Anglo-Australian Telescope to measure the redshifts of approximately 220,000 galaxies during 1995 to 2002

THIS IMAGE REPRESENTS THE DISTRIBUTION OF (conventional) MATTER IN THE LOCAL UNIVERSE



The "Millennium Simulation"
(Springel et al. 2005)

www.mpa-garching.mpg.de/galform/millennium/



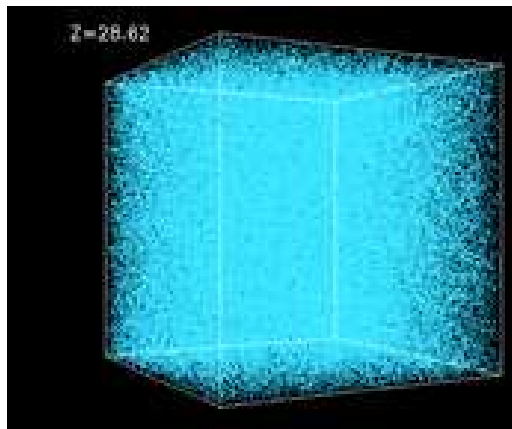
(M-)hydrocodes:

ENZO – Eulerian

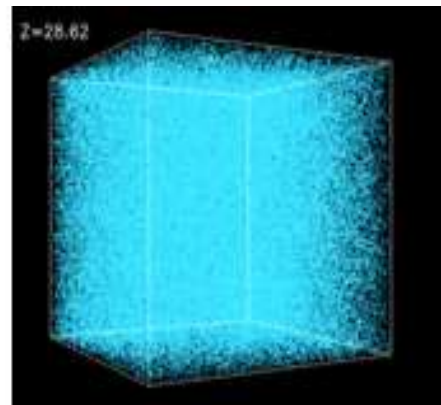
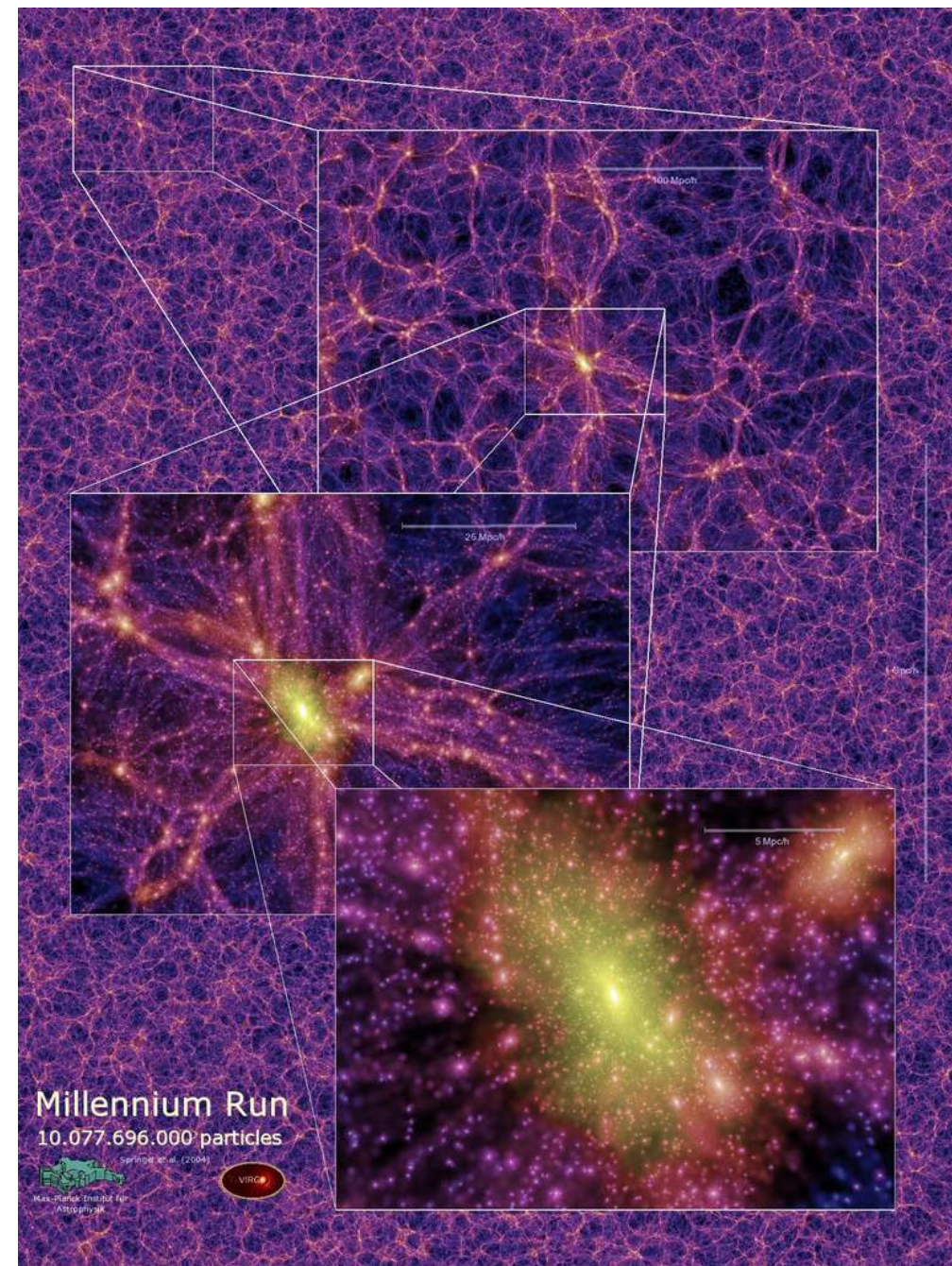
GADGET – Lagrangian

How astrophysicists
Imagine the Universe

"The Cosmic Web"



Samples of a (test)
volume of Universe



The simulation shows the formation of clusters and large-scale filaments in the Cold Dark Matter model with dark energy. Evolution of structures in a 43 Mpc box from $z=30$ to 0

Let's go back to Navier-Stokes' equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = -\frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\eta}{\rho} \nabla^2 \vec{v}$$

It can be rewritten by using the vector identity

$$\frac{1}{2} \vec{\nabla} v^2 = \vec{v} \times (\vec{\nabla} \times \vec{v}) + (\vec{v} \cdot \vec{\nabla}) \vec{v}$$

$$\frac{\partial \vec{v}}{\partial t} = -\frac{1}{2} \vec{\nabla} v^2 + \vec{v} \times (\vec{\nabla} \times \vec{v}) - \frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\eta}{\rho} \nabla^2 \vec{v}$$

Let's consider the rotational of such relation (rotational of **gradients** are 0):

$$\frac{\partial (\vec{\nabla} \times \vec{v})}{\partial t} = \vec{\nabla} \times [\vec{v} \times (\vec{\nabla} \times \vec{v})] + \frac{\eta}{\rho} \nabla^2 (\vec{\nabla} \times \vec{v})$$

Let's also introduce $\vec{\Omega} = \vec{\nabla} \times \vec{v}$ defined as **vorticity**

$$\frac{\partial \vec{\Omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\Omega}) + \frac{\eta}{\rho} \nabla^2 \vec{\Omega}$$

The quantity $\vec{\Omega}$ is known as **vorticity** and is similar to an angular velocity

$$\frac{1}{2} \oint \vec{v} \cdot d\vec{l} = \frac{1}{2} \int \int (\vec{\nabla} \times \vec{v}) \cdot \hat{n} dS = \frac{1}{2} \int \int \vec{\Omega} \cdot \hat{n} dS \approx \pi r^2 \Omega$$

If we apply the Stokes' theorem to a circumference with radius r :

but the left side is $\pi r v$ then $v = r \Omega$

Physical and numerical examples (in general with turbulence)



https://www.youtube.com/watch?v=TdZdqfD0LTI&feature=emb_rel_end

<https://www.youtube.com/watch?v=NjSFR40SY58>

<https://www.tng-project.org>

Astrophysical Jets

Illustris simulation

Illustris web site

If viscosity is negligible ($\eta = 0$) (Helmoltz equation)

$$\frac{\partial \vec{\Omega}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{\Omega})$$

Then, if initially $\Omega = 0$ it stays zero $\rightarrow$ the flux of across any given surface in motion with the fluid is constant

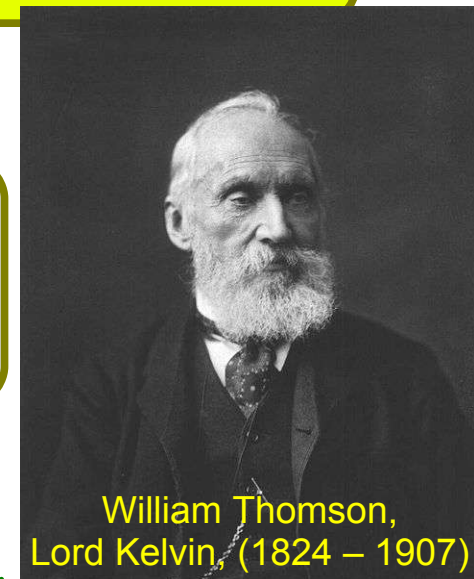
In an inviscid, barotropic flow with conservative body forces, the circulation around a closed curve moving with the fluid remains constant with time (Kelvin's theorem)

Similar to H field in a plasma

The structure of the velocity field is bound to the fluid element.
If such element changes shape during its motion, then also the velocity field is modified accordingly

Consequences....

- In an ideal fluid (viscosity is 0) vortices cannot be either created or disrupted
- Viscosity is necessary for creation/disruption of vortices
- If $\Omega = 0$ is initially zero, then it stays zero!





Hydrostatic equilibrium (1)

It represents a condition where time derivatives are 0 (i.e. **equilibrium**) and the fluid does not move ($v = 0$ i.e. **hydrostatic**).

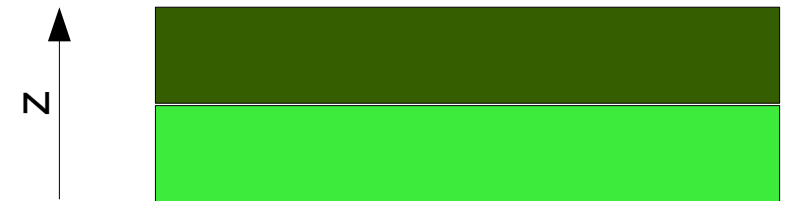
Mass and energy conservation equations are identically satisfied.

In case of an inviscid fluid, the momentum conservation becomes

$$0 = -\vec{\nabla} \phi - \frac{1}{\rho} \vec{\nabla} p \quad \text{namely} \quad \frac{1}{\rho} \vec{\nabla} p = -\vec{\nabla} \phi$$

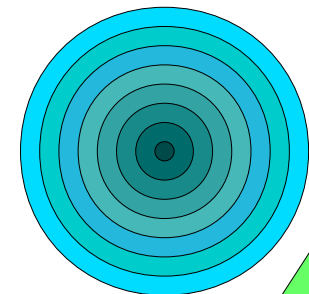
AIM: in case we know the barotropic equation of state [i.e. $p = p(\rho)$] we want to solve for the pressure / density distribution everywhere

$$p = p(r) \quad \text{and/or} \quad \rho = \rho(r)$$



There are several possible applications:

- slabs of stratified (isothermal) fluids
- (isothermal) spherical distribution



FYI,
Never asked
@ exams



Hydrostatic equilibrium (2)

For an isothermal distribution the hydrostatic equilibrium can be written in (polar) spherical coordinates, and then

$$\vec{\nabla} p = -\rho \vec{\nabla} \phi \quad \text{becomes}$$

$$\frac{dp}{dr} = -\rho \frac{d\phi}{dr}$$

i.e. surfaces with constant p, ρ, ψ coincide

In general, the barotropic equation of state

$$p = K \rho^{1 + \frac{1}{n}}$$

is known as polytrope, and can be interpreted as a power law.

More will be given in "Stellar Astrophysics".

Details in Clarke & Churchwell, Chap. 5



Let's consider a **compressible** fluid at rest ($\vec{v}_0 = 0$), with initial pressure p_0 and density ρ_0 in which **friction, gravity and magnetic forces can be neglected**.

Let's generate a small **1-D** perturbation **along the x axis** so that [small $\rightarrow \delta\rho = \rho' \ll \rho_0$]

$$\rho(x, t) = \rho_0 + \delta\rho(x, t) = \rho_0 + \rho'$$

$$p(x, t) = p_0 + \delta p(x, t) = p_0 + p'$$

$$v(x, t) = v_0 + \delta v(x, t) = v'$$

Let's substitute these relations in the 1-D eq. of motion (Euler's equation)

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x} - \frac{1}{\rho} \frac{\partial f}{\partial x} \quad (\text{No external forces, no gravity})$$

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial x} = - \frac{1}{\rho_0 + \rho'} \frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial x} = 0 \quad (1)$$

In the 1-D equation of continuity

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$$

$$\frac{\partial}{\partial t}(\rho_0 + \rho') + (\rho_0 + \rho') \frac{\partial v'}{\partial x} = 0 \quad \left[v' \frac{\partial \rho'}{\partial x} \right] \text{ neglected, since both } v' \text{ and } \frac{\partial \rho'}{\partial x} \text{ are small}$$

$$\frac{\partial \rho'}{\partial t} + \rho_0 \frac{\partial v'}{\partial x} = 0$$

$$\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \frac{\partial v'}{\partial x} = 0 \quad (2)$$

We are left with: (1), (2)

$$\frac{\partial v'}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial x} = 0$$

$$\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \frac{\partial v'}{\partial x} = 0$$

We are left with: (1), (2)

$$\frac{\partial v'}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial x} = 0$$

$$\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \frac{\partial v'}{\partial x} = 0$$

Then let's derive the first equation wrt x and the second wrt t and subtracting the second we get

$$\frac{\partial}{\partial x} \left(\frac{\partial v'}{\partial t} + \frac{1}{\rho_0} \frac{\partial p'}{\partial x} \right) = \frac{\partial^2 v'}{\partial t \partial x} + \frac{1}{\rho_0} \frac{\partial^2 p'}{\partial x^2} = 0$$

$$\frac{\partial}{\partial t} \left(\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \frac{\partial v'}{\partial x} \right) = \frac{1}{\rho_0} \frac{\partial^2 \rho'}{\partial t^2} + \frac{\partial^2 v'}{\partial x \partial t} = 0$$

$$\frac{1}{\rho_0} \frac{\partial^2 p'}{\partial x^2} - \frac{1}{\rho_0} \frac{\partial^2 \rho'}{\partial t^2} = \frac{\partial^2 p'}{\partial x^2} - \frac{\partial^2 \rho'}{\partial t^2} = 0$$

known as **dispersion relation** (sometimes its solutions are termed in that way)

We can consider that **all motions in an ideal fluid are adiabatic** to the lowest order

of approximation $p' = \delta p = \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0} \delta \rho = \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0} \rho'$

and substitute in the previous equation to get

$$\left(\frac{\partial p}{\partial \rho} \right)_{\rho_0} \frac{\partial^2 \rho'}{\partial x^2} - \frac{\partial^2 \rho'}{\partial t^2} = 0$$

which is the expression of a wave propagating along the x axis with velocity

$$c_s = \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0}^{1/2}$$

Considering the 1-D expression, **p** and **ρ** solutions have the form

$$\begin{aligned} p' = \delta p &= f_p(x - c_s t) + g_p(x + c_s t) \\ \rho' = \delta \rho &= f_\rho(x - c_s t) + g_\rho(x + c_s t) \end{aligned}$$

It can be written releasing the (1-D) starting condition as (3-D)

$$\frac{\partial^2 \rho'}{\partial t^2} - c_s^2 \nabla^2 \rho' = 0$$

solutions are

$$\frac{\rho'}{\rho_0} = A e^{i(\vec{k} \cdot \vec{r} \pm \omega t)}$$

where $\omega^2 = k^2 c_s^2$; $A \ll 1$

Longitudinal waves (displacement along the direction of propagation)

➡ Sound waves allow the propagations of perturbations in all hydrodynamical quantities (except for entropy and vorticity, which are bound to each fluid element).

In case the fluid is not ideal (i.e. viscosity & heat conduction), amplitude of sound waves will be damped and their energy dissipated as heat (e.g. Padmanabhan, p 386-387)

We DO NOT enter into this business!

FYI,
not for study



$$c_s = \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0}^{1/2} \quad \text{is the sound speed in an ideal gas}$$

If the perturbation can be assumed to be **adiabatic** (like in most cases for an ideal gas)

$$p \approx \rho^\Gamma \quad c_s^2 = \Gamma \frac{p}{\rho} = \Gamma \rho^{\Gamma-1} = \Gamma \frac{kT}{\mu}$$

in fact $pV^\Gamma = \text{const}$ namely $p = p_0 (r/r_0)^\Gamma$ then $(p \propto r^\Gamma)$

$$c_s = \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0}^{1/2} = \left[\Gamma \left(\frac{p_0}{\rho_0^\Gamma} \right) \rho^{\Gamma-1} \right]^{1/2} = \left(\Gamma \frac{p}{\rho} \right)^{1/2} = (\Gamma \rho^{\Gamma-1})^{1/2} = \left(\Gamma \frac{kT}{\mu} \right)^{1/2}$$

$[\mu = \text{mean molecular (atomic) mass}]$

In case of an **isothermal** perturbation (then $P \propto \rho$)

$$c_s^2 = \frac{P}{\rho} = \frac{kT}{\mu} \quad [\mu = \text{mean molecular (atomic) mass}]$$

In fact

$$c_s = \left(\frac{\partial p}{\partial \rho} \right)_{\rho_0}^{1/2} = \left(\frac{p}{\rho} \right)^{1/2} = \left(\frac{kT}{\mu} \right)^{1/2}$$

In general a fluid in motion is described by

$$M = \frac{v}{c_s}$$

known as **Mach number**.

In case either $M < 1$ or $M > 1$ the motion is termed sub/super-sonic



Test: values for the fluid **Earth atmosphere**

$$\begin{aligned} p_0 &= 1.013250 \cdot 10^5 \text{ Nm}^{-2} ; \rho_0 = 1.2928 \text{ kg m}^{-3} ; k = 1.38 \cdot 10^{-23} \text{ JK}^{-1} ; \\ \mu &\sim 0.029 \text{ kg m}^{-3} ; N = 6.022 \cdot 10^{23} \text{ mol}^{-1} ; (\Gamma = 1.4) \end{aligned}$$



Let's derive the sound speed in a gas made of atomic hydrogen

$$\Gamma = 4/3 = 1.33 \quad \mu = 1.67 \times 10^{-27} \text{ kg} \quad k = 1.38 \times 10^{-23} \text{ J K}^{-1}$$

$$c_s = \sqrt{\Gamma \frac{kT}{\mu}} = \sqrt{1.33 \frac{1.38 \times 10^{-23}}{1.67 \times 10^{-27}} \cdot T \text{ [J K}^{-1} \text{ kg}^{-1}]} = 1.05 \times 10^2 \cdot \sqrt{T} \text{ [m/s]}$$

$$T = 10^2 \text{ K} \quad (\text{e.g. condensed cloud, partly atomic/molecular}) \quad c_s \approx 1000 \text{ m/s} = 1 \text{ km/s}$$

$$T = 10^3 \text{ K} \quad (\text{e.g. typical ISM material in the disk of spiral galaxies}) \quad c_s \approx 3000 \text{ m/s} = 3 \text{ km/s}$$

$$T = 10^4 \text{ K} \quad (\text{e.g. HII regions, see later on for explanation}) \quad c_s \approx 10000 \text{ m/s} = 10 \text{ km/s}$$

$$T = 10^6 \text{ K} \quad (\text{e.g. Bulge of spiral galaxies, Ellipticals}) \quad c_s \approx 100000 \text{ m/s} = 100 \text{ km/s}$$

$$T = 10^8 \text{ K} \quad (\text{e.g. Intergalactic medium in massive galaxy clusters}) \quad c_s \approx 1000000 \text{ m/s} = 1000 \text{ km/s}$$



A fluid is considered **incompressible** when (space and time variations are negligible)

$$\frac{\Delta \rho}{\rho} \ll 1$$

Neglecting all forces except pressure, let's consider

$$\Delta \rho \simeq \left(\frac{\partial \rho}{\partial p} \right) \Delta p = \frac{1}{c_s^2} \Delta p$$

Using Euler's equation in its linear approximation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} = - \frac{1}{\rho} \vec{\nabla} p$$

$$\frac{v}{\tau} + \frac{v^2}{L} \left(= \frac{2v^2}{L} \right) \approx - \frac{1}{\rho} \frac{p}{L}$$

[remember that $\frac{1}{\tau} \approx \frac{v}{L}$]

namely $2v^2 \rho \approx -p$ and considering $\Delta p \approx p/2$ we get $\Delta p \approx -\rho v^2$
which can be entered into the $\Delta \rho$ expression

$$\Delta \rho \simeq \frac{1}{c_s^2} \Delta p = \frac{1}{c_s^2} \rho v^2 \rightarrow \frac{\Delta \rho}{\rho} \simeq \frac{v^2}{c_s^2} = M^2$$

FYI,
Never asked
@ exams



Then for having an incompressible fluid, it is necessary that its motion is subsonic:

$$M^2 \ll 1.$$

In practice, in all cases when
namely $\Delta\rho / \rho \leq 0.1 - 0.2$
 $(v \leq 0.4 c_s)$

FYI,
Never asked
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the fluid can be considered **incompressible**.

Alternatively taking

$$\psi = f(x - c_s t)$$

the velocity along the x axis is

$$v' = \frac{\partial \psi}{\partial t} = f'$$

and it comes out that the pressure is

$$p' = -\rho \frac{\partial \psi}{\partial t} = c_s \frac{\rho'}{\rho}$$

from the def of sound speed

$$p' = c_s^2 \rho'$$

and finally

$$v' = c_s \left(\frac{\rho'}{\rho} \right)$$

FYI,
not for studyc

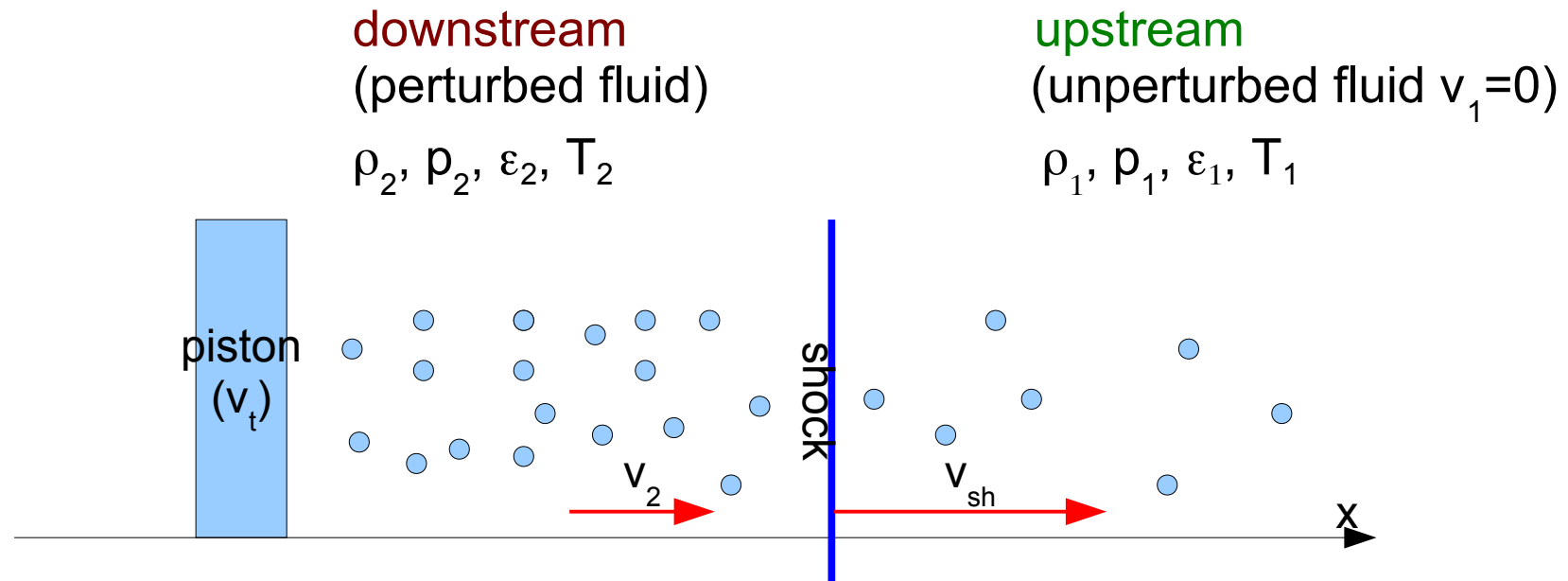
Shock waves are **not reversible** discontinuities in the properties of a fluid when a body (solid, another fluid, ...) is propagating within that fluid faster than the sound speed..

In the ideal case:

- LTE is supposed to hold everywhere except in the shock front ($\partial/\partial x \neq 0$)
- a stationary regime is considered ($\partial/\partial t = 0$)

➡ In Astrophysics the aim is to search for relationships between perturbed and unperturbed parameters



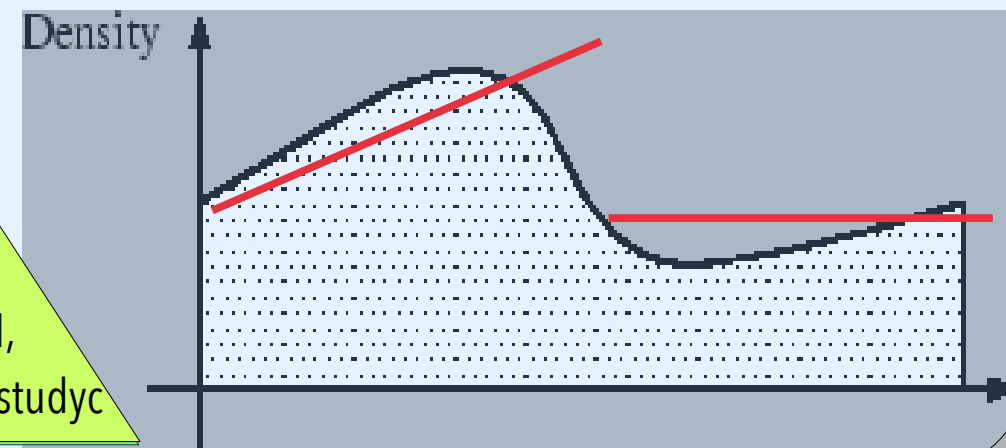


a shock is formed when a perturbation is moving with a velocity larger than the sound speed pushing/compressing the fluid encountered during its motion

also adiabatic sound waves can grow to shock waves
[c_s is higher where ρ, T are higher, leaving the denser regions progressively move at higher velocities (c_s continuously grows!) until a jump is formed]

the shock compresses, heats and drags the shocked material

FYI,
not for studyc





It is possible to define a (comoving) surface where physical quantities **abruptly** change

It is the surface of the shock, where discontinuities are manifested

In that RF all the HD equation must hold

⇒ simplest case: they have a stationary form ($\partial/\partial t = 0$), and happen along a given direction, i.e. x

$$\frac{\partial \rho v_x}{\partial x} = 0$$

$$\frac{\partial}{\partial x} (\rho v_x^2 + p) = 0; \quad \frac{\partial}{\partial x} (\rho v_x v_y) = 0; \quad \frac{\partial}{\partial x} (\rho v_x v_z) = 0$$

$$\frac{\partial}{\partial x} \left[\rho v_x \left(w + \frac{v^2}{2} \right) \right] = 0$$

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Case 1: Any mass can't cross the surface $\rho_1 v_{1x} = \rho_2 v_{2x} = 0$

Density can't be 0, consequently all velocities are 0;

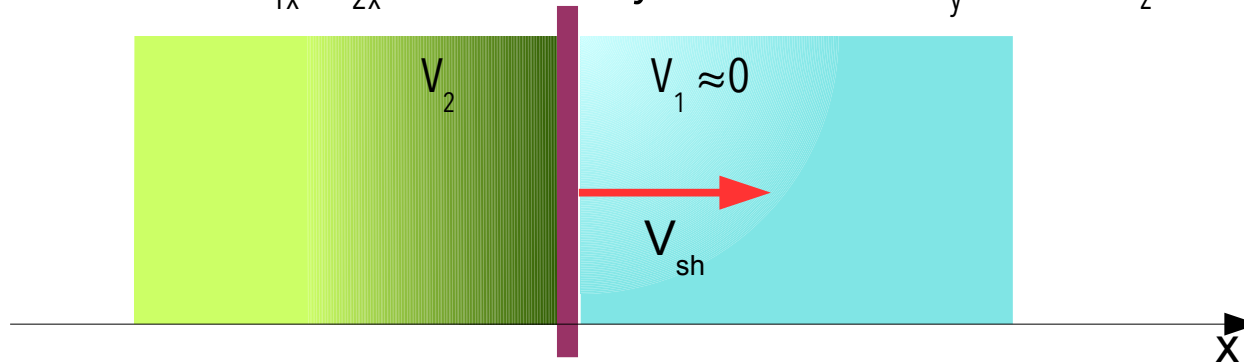
from the second equation $\frac{\partial p}{\partial x} = 0$ and then only ρ, v_y, v_z can be discontinuous

(tangential discontinuity)

It is possible to show that such tangential discontinuities are unstable (grow). In this case the two fluids will be mixed by turbulence



➔ **Case 2:** In general (= all the other cases) the mass flux across the surface $\frac{\partial \rho v_x}{\partial x}$ is not 0 and v_{1x}, v_{2x} are not 0 anymore while v_y and v_z become continuous.



Let's choose a RF [' quantities] on the shock front: unperturbed fluid falls with $v'_1 = -v_{sh}$

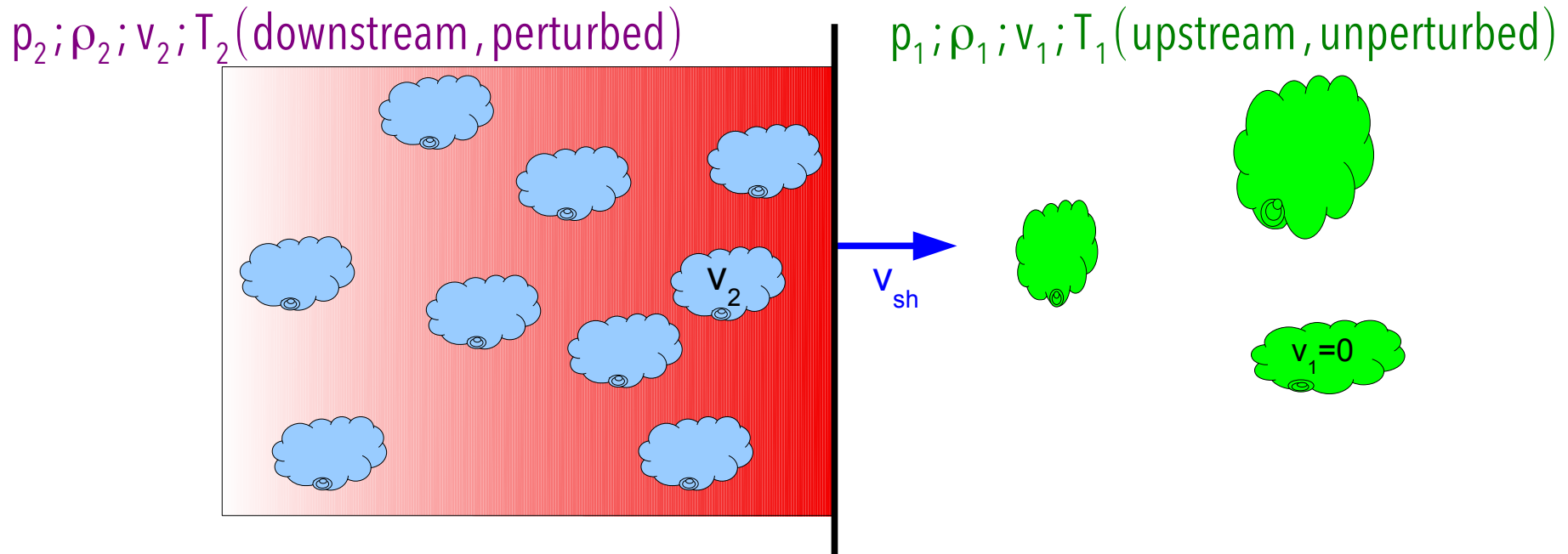
The shocked matter is dropped behind at $v'_2 = v_2 - v_{sh}$

An external observer measures $v_1 = 0$, v_2 and $v_{sh} > v_2$

Let's write the conservation laws for HD fluids on the discontinuity surface:

$$(1) \quad \rho_1 v'_1 = \rho_2 v'_2 \quad \text{i.e.} \quad \frac{\rho_1}{\rho_2} = \frac{v'_2}{v'_1}$$

$$(2) \quad \rho_1 v'^2_1 + p_1 = \rho_2 v'^2_2 + p_2 \quad \text{i.e.} \quad \rho_1 v'^2_1 - \rho_2 v'^2_2 = p_2 - p_1$$



$$(3) \quad \rho_1 v'_1 \left(\frac{v_1'^2}{2} + \underbrace{\epsilon_1 + \frac{p_1}{\rho_1}}_{w_1} \right) = \rho_2 v'_2 \left(\frac{v_2'^2}{2} + \underbrace{\epsilon_2 + \frac{p_2}{\rho_2}}_{w_2} \right)$$

FYI,
Never asked
@ exams

where ϵ and $w = \epsilon + p/\rho$ are the energy and is the enthalpy per unit mass.

➡ We have just written that on the shock surface the INCOMING FLUID can't stop and MUST FLOW, preserving mass, momentum and energy (nothing can accumulate on the shock surface)

Then p_2 and v'_2 can be obtained; finally after some algebra (not reported!) we get

$$\frac{\rho_2}{\rho_1} = \frac{(\Gamma + 1)M^2}{2 + (\Gamma - 1)M^2} = \frac{v'_1}{v'_2}$$

Once ρ_2 and v'_2 are known, also the pressure ratio can be defined

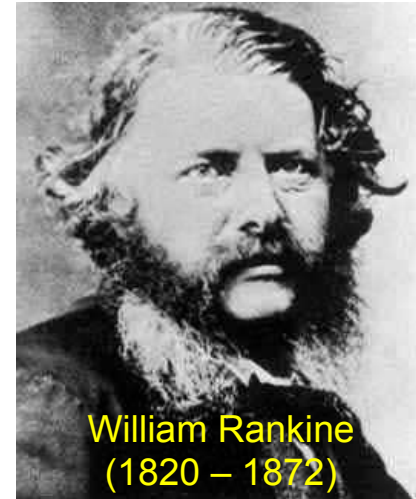
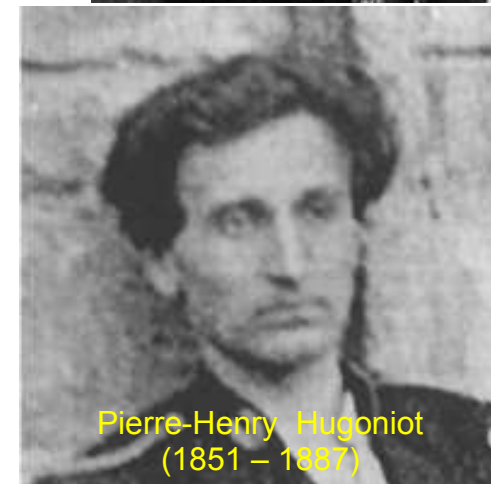
$$\frac{p_2}{p_1} = \frac{2\Gamma M^2 - (\Gamma - 1)}{(\Gamma + 1)}$$

known as **Rankine - Hugoniot** relationships in the shock RF.

From the first equation

$$\frac{v'_2}{v'_1} = \frac{2 + (\Gamma - 1)M^2}{(\Gamma + 1)M^2} \left\{ \begin{array}{l} = \frac{1}{4} \text{ if monoatomic gas, } M \rightarrow \infty \\ = 1 \text{ if } M \rightarrow 1 \end{array} \right.$$

In the observer's frame, with $M \gg 1$:

$$v_2 = (v_{sh} + v'_2) = \frac{3}{4}v_{sh}$$


Strong (adiabatic) shock waves : in case of $M \gg 1$

$$\frac{v'_1}{v'_2} = \frac{\rho_2}{\rho_1} = \frac{(\Gamma + 1)M^2}{2 + (\Gamma - 1)M^2} \simeq \frac{(\Gamma + 1)}{(\Gamma - 1)}$$

Independent of M

$$\frac{p_2}{p_1} = \frac{2\Gamma M^2 - (\Gamma - 1)}{(\Gamma + 1)} \simeq \frac{2\Gamma}{(\Gamma + 1)} \cdot M^2$$

Can grow to large p

since $PV \propto T$ then $\frac{P}{\rho} \propto T$ and we can constrain the jump in temperature:

$$\frac{T_2}{T_1} = \frac{p_2}{p_1} \cdot \frac{\rho_1}{\rho_2} \simeq \frac{2\Gamma}{(\Gamma + 1)} \cdot M^2 \cdot \frac{\Gamma - 1}{\Gamma + 1} = \frac{2\Gamma(\Gamma - 1)}{(\Gamma + 1)^2} \cdot M^2$$

Can grow to large T

→ The temperature (and pressure) of the shocked material can grow without limitations.

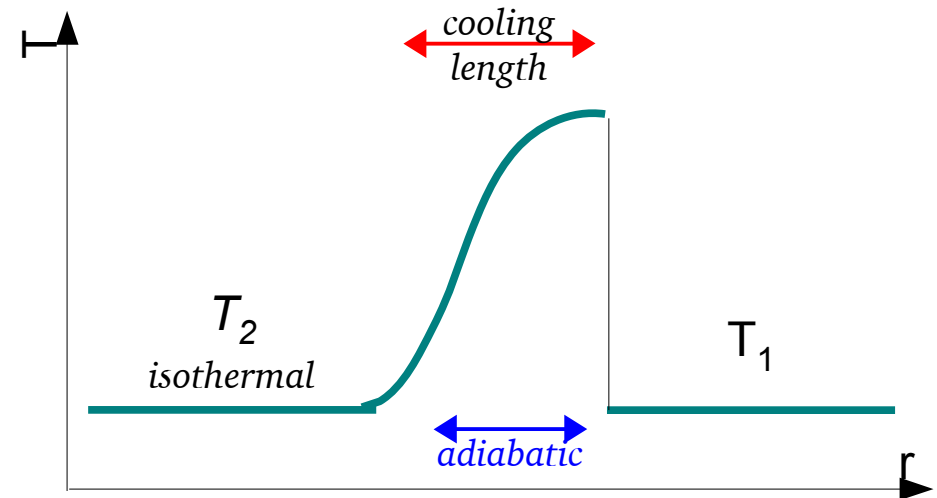
Isothermal shock waves:

- the gas is capable to rapidly radiate the thermal energy acquired crossing the shock surface (the cooling time is extremely short) then $T_2 = T_1$

It is equivalent to have $\Gamma = 1$ in all the earlier equations.

$$\frac{\rho_2}{\rho_1} = M^2$$

$$\frac{p_2}{p_1} = M^2$$



The fluid, then, can be highly compressed.

The shocked fluid can be considered isothermal if the energy transferred by the shock to the ambient gas is nearly immediately dissipated (e.g. via radiation/expansion)



details: either chap 7.2 cowie & carswell or chap 8.11 Padmanaban or chap. 8 Fanti & Fanti



* The shock converts kinetic into internal (thermal) energy

* **Thickness**: is of the order of the mean free path λ ; from the relation $\lambda n \sigma = 1$

$$\lambda = 1/n\sigma$$

where n =numeric density; σ =cross section of the process

* The knowledge of the parameters of the unperturbed/perturbed medium (pex 1/2) makes therefore possible to derive those of the shocked/unshocked material

Also entropy is discontinuous on the shock surface $s_2 \neq s_1$ and then $s_2 > s_1$ while it is preserved upstream and downstream. The entropy increase generated by collisions among fluid particles

For reference to λ :

electronic radius $5.3 \times 10^{-11} \text{ m}$, $n_{\text{ISM}} \sim 1 \text{ cm}^{-3}$, $1 \text{ pc} = 3.09 \times 10^{16} \text{ m}$, $UA = 1.5 \times 10^{11} \text{ m}$

The shock converts kinetic to internal (thermal) energy

The case of SNR:

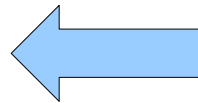
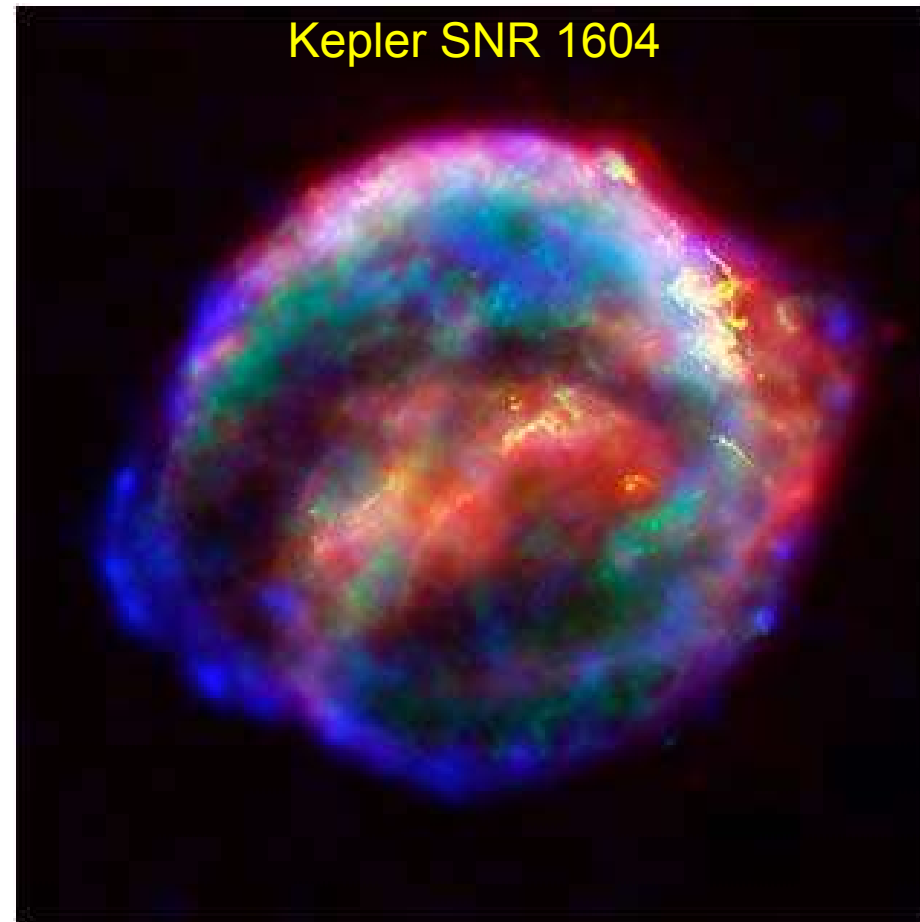
$$n_1 \sim 1 - 10 \text{ cm}^{-3}$$

$$T \sim 1000 \text{ K};$$

$$\mu = m_H \sim 1.7 \times 10^{-27} \text{ kg};$$

$$c_s = (\Gamma kT/\mu)^{1/2}$$

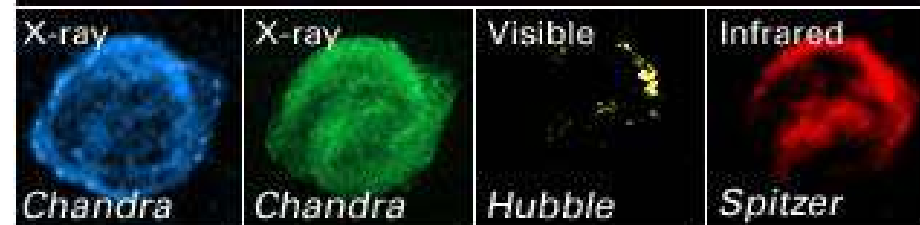
$$v_{\text{SN}} \sim 10^4 \text{ km/s};$$



Kepler SNR 1604

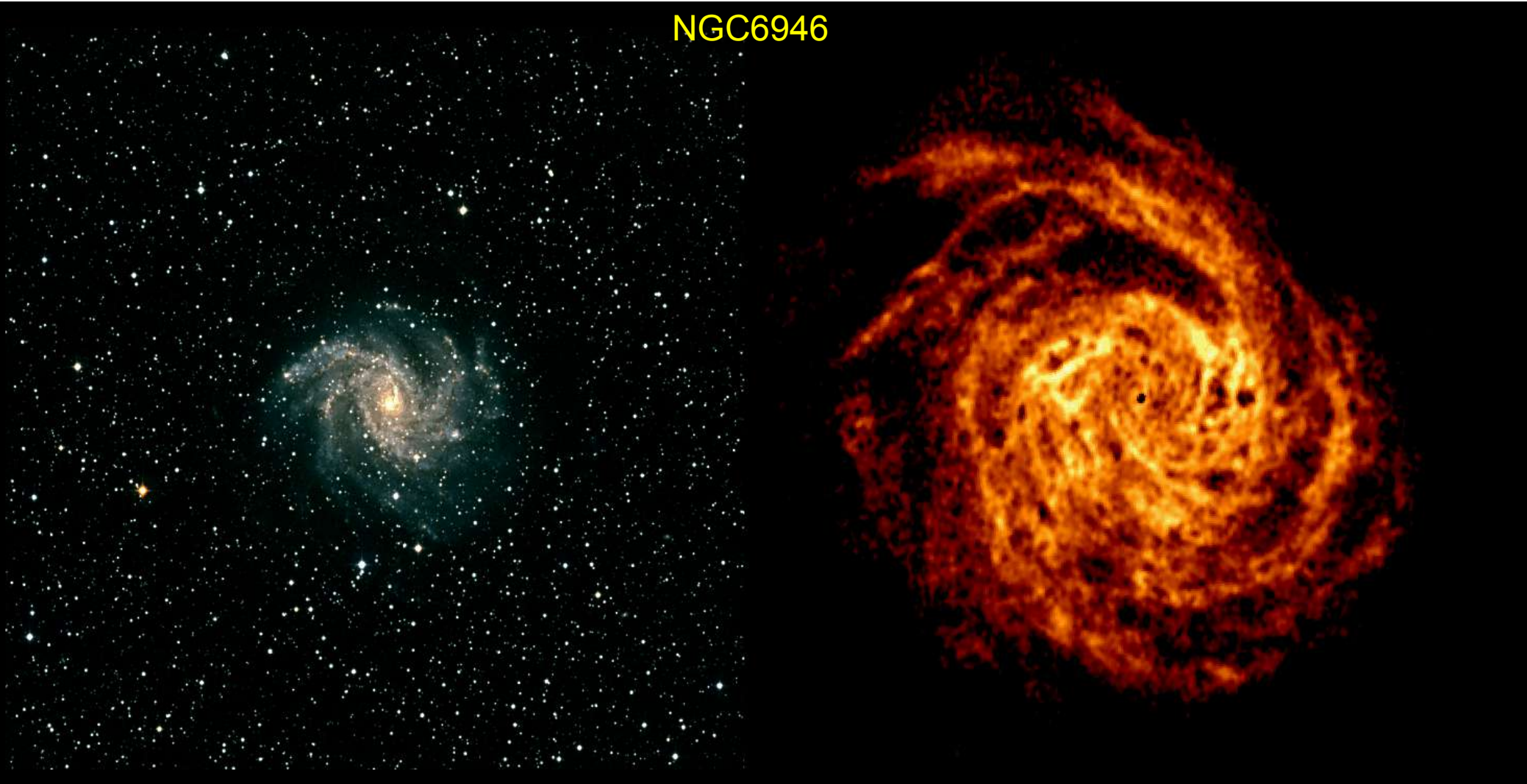
The star envelope expands at $M \gg 1$

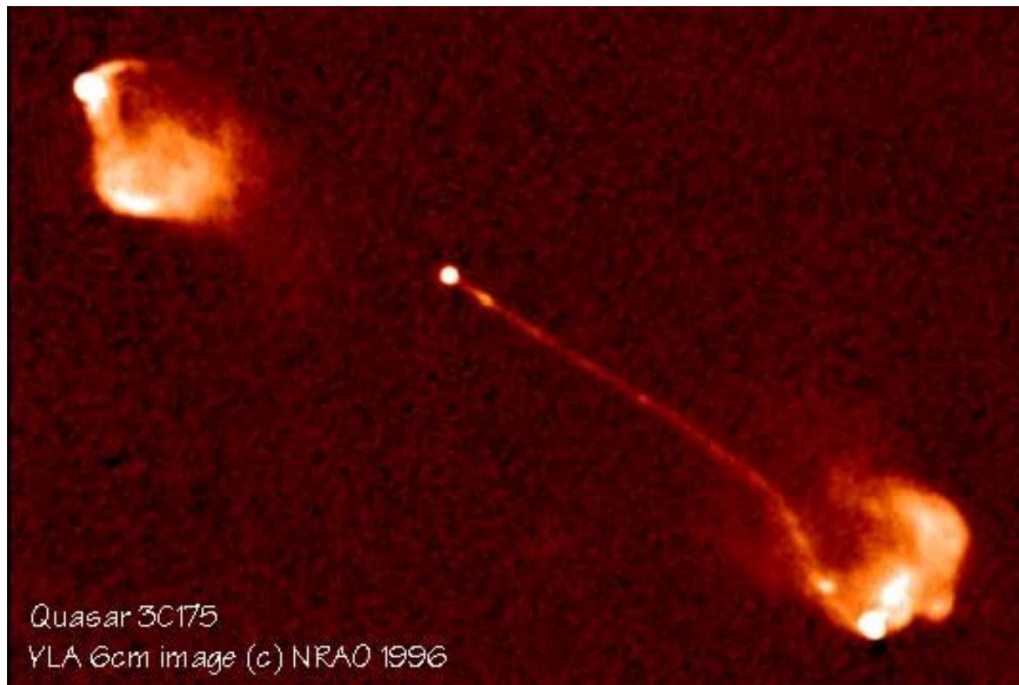
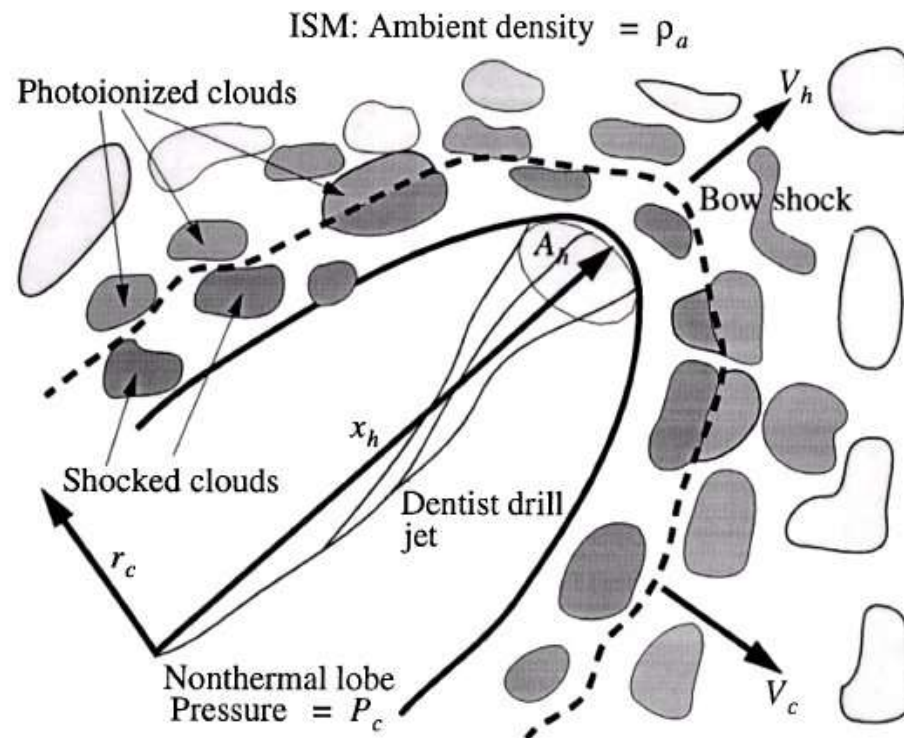
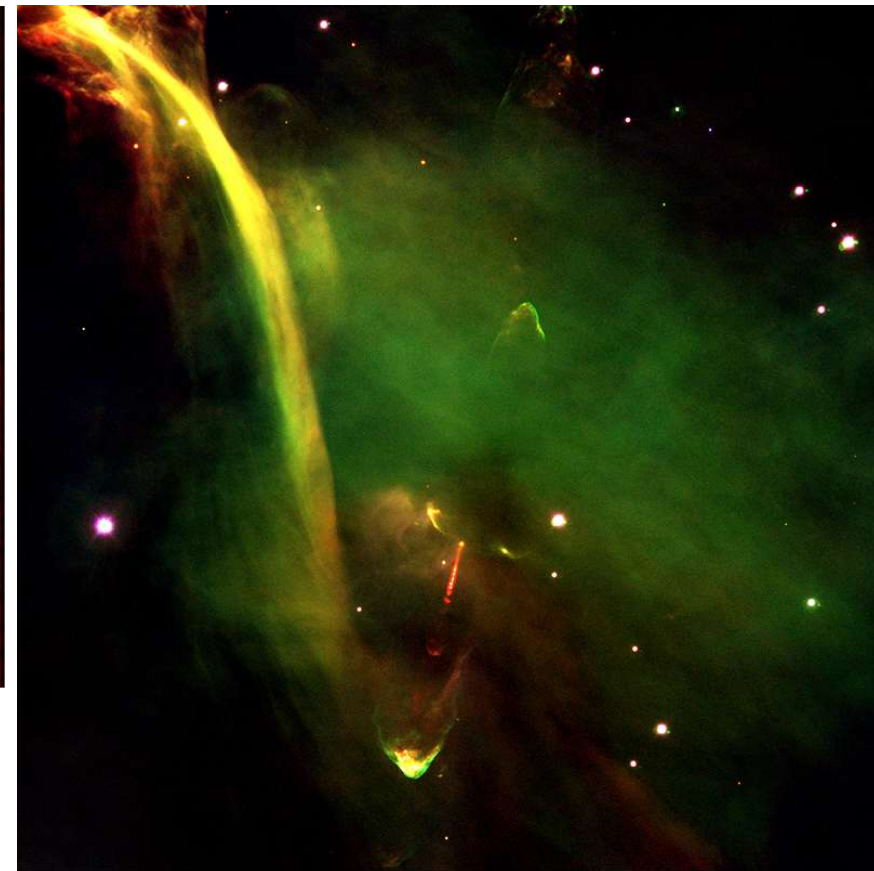
Supernova explosions eject plasma at
 $V \sim 10^4 \text{ km s}^{-1}$ in the Circum Stellar Medium



SNR explosions and consequences on ρ_2 and on the ambient medium (bubbles!)

NGC6946





Protostar HH-34 in Orion (VLT KUEYEN + FORS2)

ESO PR Photo 40b/99 (17 November 1999)

© European Southern Observatory



Magnetic fields become important in case of substantial electron (ion) density
i.e. $T > 10^4 \text{ K}$ [$\sim 10^6 - 10^9 \text{ K}$ are often typically found in astrophysics]

An ionized fluid is known as **plasma** which can be treated as an ideal fluid when the linear size of a phenomenon/body exceeds the Debye length

- [1. random kinetic energy of particles is much larger than the electrostatic potential energy between two particles
- 2. the E field of a single particle is nulled by the field of its neighboring charges with opposite sign
- 3. charge density $\rho_e \sim 0$]

$$\lambda_D \equiv \left(\frac{kT}{4\pi n_e e^2} \right)^{1/2} = 6.9 \text{ cm} \left(\frac{T}{^\circ\text{K}} \right)^{1/2} \left(\frac{n_e}{\text{cm}^{-3}} \right)^{-1/2}$$



An **astrophysical plasma** must satisfy:

- 1. fluid velocity is small ($v \ll c$, and all the terms v/c^2 can be neglected)
- 2. electric conductivity is large ($\sigma \rightarrow \infty$)
- 3. linear scale $\ell \gg \lambda_D$
- 4. collisions are ruling the plasma physics ($v_c > v_L > v_{\text{phenomenon}}$)

i.e. **The size of the fluid exceeds the Larmor radius which is, in turn, larger than the mean free path between collisions**

Typical orders of magnitude (numbers should be interpreted as 10^x)

Celestial body	L(cm)	T (K)	σ (1/sec)	τ (sec)	H (Gauss)
Solar corona	11	6	16	18	5
HII region	18	4	13	28	-5
WD	8	6	16	12	8
NS	6	9	22	14	13
ISGas	20	2	11	28	-6
IGGas	23	8	19	44	-7
Lab. Plasma	2	4	13	-3	3

$$\sigma = \frac{n_e e^2 \lambda}{2 m_e v} = \frac{10^8 T^{3/2}}{g_{FF}} \text{ sec}^{-1}$$

$$\tau(H) = \frac{4\pi}{c^2} \sigma L^2 \text{ sec}$$

λ is the mean free path

Where $\tau(H)$ is a timescale for "diffusion" of a magnetic field H , wait for a few slides!

Maxwell equations and their HD approximation;

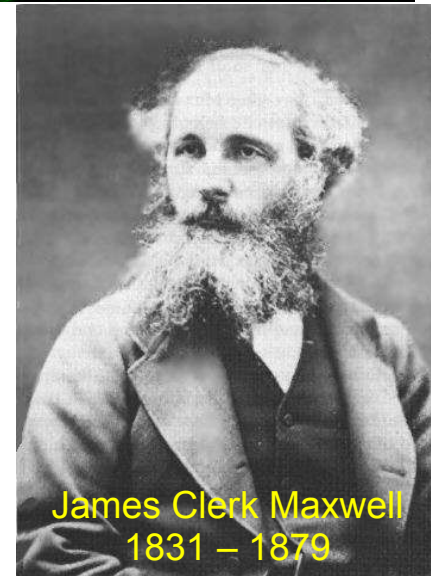
$$\nabla \cdot \vec{H} = 0$$

$$\nabla \cdot \vec{E} = 4\pi\rho_e$$

$$\rightarrow \nabla \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

Displacement
current



Let's consider the divergence of the third equation:

$$\rightarrow 0 = \frac{4\pi}{c} \nabla \cdot \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} \nabla \cdot \vec{E} = 0 = \frac{4\pi}{c} \nabla \cdot \vec{j} + \frac{1}{c} \frac{\partial}{\partial t} 4\pi\rho_e$$

$$0 = \frac{\partial \rho_e}{\partial t} + \nabla \cdot (\rho_e \vec{v})$$

Can interpreted as **charge conservation** $\rightarrow$ [$\approx$ the equation of continuity in HD, which remains unchanged]

Let's compare the two contributions in equation (1)

$$\frac{4\pi}{c} \vec{j} \Leftrightarrow \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \quad \text{Ohm's law} \quad \vec{E} = \frac{\vec{j}}{\sigma}$$

Let's derive $(1/c)$ Ohm's law

$$\frac{1}{c} \frac{\partial \vec{E}}{\partial t} = \frac{1}{c\sigma} \frac{\partial \vec{j}}{\partial t} \approx [\text{orders of magnitude}] \approx \frac{1}{c\sigma} \frac{j}{T} \approx \frac{1}{c} \frac{jv}{\sigma L}$$

it turns out that

$$\frac{4\pi}{c} |\vec{j}| \gg \frac{1}{c} \frac{jv}{\sigma L} \quad 4\pi \gg \frac{v}{\sigma L}$$

In (nearly) all astrophysical cases, given that ϑ and L are generally large.

Consequence:

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \approx \frac{4\pi}{c} \vec{j}$$

Let's take the divergence:

$$\nabla \cdot (\nabla \times \vec{H}) = \frac{4\pi}{c} \nabla \cdot \vec{j} = 0 \quad \rightarrow (\text{Ohm's law}) \quad \nabla \cdot \vec{E} = 0$$

Namely $\rho_e = 0 \rightarrow$ Neutral plasma!

Fields in observer's ($\vec{E}, \vec{H}$) and rest ($\vec{E}', \vec{H}'$) frames of a plasma moving at $v \ll c$

$$\vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{H}$$

Lorentz force on a moving charge

$$\vec{H}' = \vec{H} - \frac{\vec{v}}{c} \times \vec{E}$$

$$\vec{j}' = \vec{j} - \rho_e \vec{v} \approx \vec{j}$$

$$\rho_e' = \rho_e - \frac{1}{c^2} (\vec{v} \cdot \vec{j}) \approx \rho_e \approx 0$$

Let's take the first equation:

$$\Rightarrow \vec{E} = \vec{E}' - \frac{\vec{v}}{c} \times \vec{H} = \frac{\vec{j}}{\sigma} - \frac{\vec{v}}{c} \times \vec{H}$$

$$= \frac{c}{4\pi\sigma} (\nabla \times \vec{H}) - \frac{\vec{v}}{c} \times \vec{H} \quad [\text{linearization!}]$$

$$\approx \frac{c}{4\pi\sigma} \frac{H}{L} - \frac{v}{c} H = -\frac{v}{c} H \left(1 - \frac{c^2}{4\pi\sigma v L} \right) = -\frac{v}{c} H \left(1 - \frac{1}{R_m} \right)$$



Hendrik Lorentz
1853 – 1928

Magnetic Reynolds' number $R_m = \frac{4\pi\sigma vL}{c^2} \gg 1$

It is generally true in astrophysics.

In most cases R_m is $\sim 10^6$ but often it is much larger

Consequence: (... in general ...) $E \approx -\frac{v}{c}H$

astrophysical electric fields are extremely weak (~ 0) and can be neglected;
induced fields from moving charges are generally negligible (except....).

Furthermore

$$\frac{c^2}{4\pi\sigma} = \eta_m \rightarrow \text{magnetic viscosity}$$

$$R_m = \frac{vL}{\eta_m}$$

Summary of MHD approximations:

$$\nabla \cdot \vec{H} = 0$$

$$\nabla \cdot \vec{E} = 4\pi\rho_e \approx 0$$

$$\nabla \times \vec{H} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t} \approx \frac{4\pi}{c} \vec{j}$$

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

$$\vec{E}' = \vec{E} + \frac{\vec{v}}{c} \times \vec{H} \approx 0$$

$$\vec{H}' = \vec{H} - \frac{\vec{v}}{c} \times \vec{E} \approx \vec{H}$$

$$\vec{j}' = \vec{j} - \rho_e \vec{v} \approx \vec{j} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H} \right)$$

Maxwell equations and their HD approximation

$$\nabla \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

$$\frac{\partial \vec{H}}{\partial t} = -c \vec{\nabla} \times \vec{E}$$

$$\vec{j} = \sigma \left(\vec{E} + \frac{\vec{v}}{c} \times \vec{H} \right)$$

$$\frac{\partial \vec{H}}{\partial t} = -c \vec{\nabla} \times \left(\frac{\vec{j}}{\sigma} - \frac{\vec{v}}{c} \times \vec{H} \right)$$

$$\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \vec{H}$$

FYI,
Never asked
@ exams

$$\frac{\partial \vec{H}}{\partial t} = -c \vec{\nabla} \times \left(\frac{c}{4\pi\sigma} \vec{\nabla} \times \vec{H} - \frac{\vec{v}}{c} \times \vec{H} \right)$$

$$\approx \vec{\nabla} \times (\vec{v} \times \vec{H}) - \frac{c^2}{4\pi\sigma} \vec{\nabla} \times (\vec{\nabla} \times \vec{H}) \quad \text{but} \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{H}) - \nabla^2 \vec{H}$$

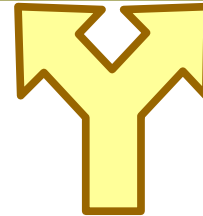
$$= \vec{\nabla} \times (\vec{v} \times \vec{H}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{H} = \vec{\nabla} \times (\vec{v} \times \vec{H}) + \eta_m \nabla^2 \vec{H}$$

$$\approx \frac{vH}{L} + \frac{c^2}{4\pi\sigma L^2} H = \frac{vH}{L} \left(1 + \frac{1}{R_m} \right)$$

$$\frac{\partial \vec{H}}{\partial t} \approx \frac{vH}{L} \left(1 + \frac{1}{R_m} \right)$$

There are two limiting cases:

$$\begin{aligned}\frac{\partial \vec{H}}{\partial t} &= \vec{\nabla} \times (\vec{v} \times \vec{H}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{H} = \\ &= \vec{\nabla} \times (\vec{v} \times \vec{H}) + \eta_m \nabla^2 \vec{H} \simeq \left(\frac{\nu H}{L} + \frac{c^2 H}{4\pi\sigma L^2} \right) = \frac{\nu H}{L} \left(1 + \frac{c^2}{4\pi\sigma \nu L} \right)\end{aligned}$$



as based on the comparison among the two terms.

The outcomes are very different!

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{v} \times \vec{H}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{H} \simeq \frac{vH}{L} + \frac{c^2}{4\pi\sigma} \frac{H}{L^2}$$

There are two limiting cases: 1. - **H field diffusion** ($\vec{v} = 0$)

$$\frac{\partial \vec{H}}{\partial t} = \frac{c^2}{4\pi\sigma} \nabla^2 \vec{H} \approx - \frac{c^2}{4\pi\sigma} \frac{H}{L^2} \quad \text{since} \quad \nabla \times (\vec{v} \times \vec{H}) = 0$$

$$\frac{\partial \vec{H}}{H} \approx - \frac{c^2}{4\pi\sigma} \frac{\partial t}{L^2}$$

$$\rightarrow H \simeq H_0 e^{-t/\tau} \quad \tau = \frac{L^2 4\pi\sigma}{c^2} \quad [\text{diffusion time}]$$

i.e. the field decays and fades away into the ambient medium

exercise: compute τ in a number of astrophysical cases: ISM (HII region, bulge corona), WD,...

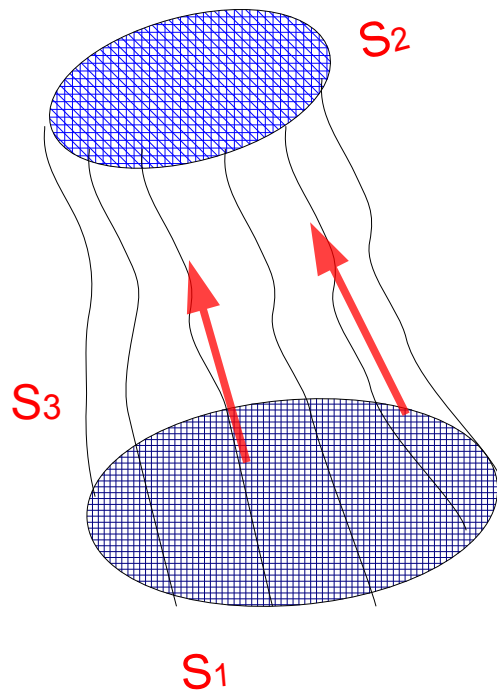
examples: ~ 1 sec for a copper sphere with radius 0.01 m; 10^{6-7} yr for NS, $\sim 10^{10}$ yr for the Sun.

2. **H frozen in matter** ($\mathbf{v} \neq \mathbf{0}$; $\mu \rightarrow \infty$) then the second term goes to 0

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{v} \times \vec{H}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{H} \simeq \frac{vH}{L} + \frac{c^2}{4\pi\sigma} \frac{H}{L^2}$$

Let's consider a closed curve defining a surface S_1 moving to another closed surface defining a surface S_2 (different from S_1) at a subsequent time dt .

The variation of the flux of H across each surface is:



$$\begin{aligned} \frac{d}{dt} \iint \vec{H} \cdot \vec{n} \, dS &\simeq \\ &\simeq \frac{1}{dt} \left[\iint_{S_2} \vec{H}(t+dt) \cdot \vec{n} \, dS_2 - \iint_{S_1} \vec{H}(t) \cdot \vec{n} \, dS_1 \right] \\ &\simeq \frac{1}{dt} \left[\iint_{S_2} \vec{H}(t) \cdot \vec{n} \, dS_2 + \iint_{S_3} \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} \, dS - \iint_{S_1} \vec{H}(t) \cdot \vec{n} \, dS_1 \right] \\ &\simeq \iint_{S_3} \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} \, dS + \frac{1}{dt} \left[\iint_{S_2} \vec{H}(t) \cdot \vec{n} \, dS_2 - \iint_{S_1} \vec{H}(t) \cdot \vec{n} \, dS_1 \right] \\ \text{if } S_t &= S_1 + S_2 + S_3 \\ &\simeq \iint_{S_3} \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} \, dS + \frac{1}{dt} \left[\iint_{S_t} \vec{H}(t) \cdot \vec{n} \, dS_t - \iint_{S_3} \vec{H}(t) \cdot \vec{n} \, dS_3 \right] \end{aligned}$$

$$\begin{aligned} \iint_s \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} dS + \frac{1}{dt} \left[\iint_{S_t} \vec{H}(t) \cdot \vec{n} dS_t - \iint_{S_3} \vec{H}(t) \cdot \vec{n} dS_3 \right] = \\ = \iint_s \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} dS + \frac{1}{dt} \left[\iiint_v \nabla \cdot \vec{H}(t) \cdot \vec{n} dV - \iint_{S_3} \vec{H}(t) \cdot \vec{n} dS_3 \right] = \end{aligned}$$

but $\vec{n} \cdot d\vec{S}|_{S_3} = d\vec{l} \times \vec{v} dt$

$$= \iint_s \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} dS - \frac{1}{dt} \oint_{S_3} \vec{H}(t) \cdot d\vec{l} \times \vec{v} dt =$$

$$= \iint_s \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} dS - \oint_{S_3} \vec{H}(t) \cdot (d\vec{l} \times \vec{v}) =$$

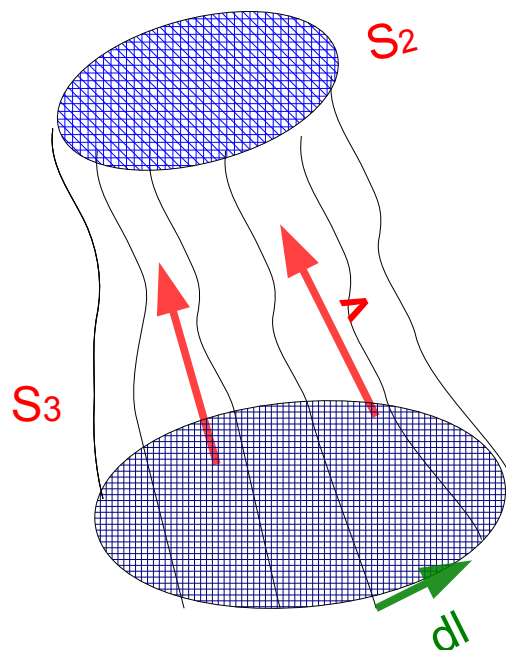
$$= \iint_s \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} dS - \oint_{S_3} (\vec{v} \times \vec{H}(t)) \cdot d\vec{l} =$$

$$= \iint_s \left(\frac{\partial \vec{H}}{\partial t} \right) \cdot \vec{n} dS - \iint_s \nabla \times (\vec{v} \times \vec{H}) \cdot \vec{n} dS =$$

$$= \iint_s \left[\frac{\partial \vec{H}}{\partial t} - \nabla \times (\vec{v} \times \vec{H}) \right] \cdot \vec{n} dS =$$

$$S_1 = \iint \left(\frac{c^2}{4\pi\sigma} \nabla^2 \vec{H} \right) dS \rightarrow 0 \quad [\sigma \rightarrow \infty]$$

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{v} \times \vec{H}) = 0 \text{ implies magnetic flux conservation}$$



FYI,
Never asked
@ exams

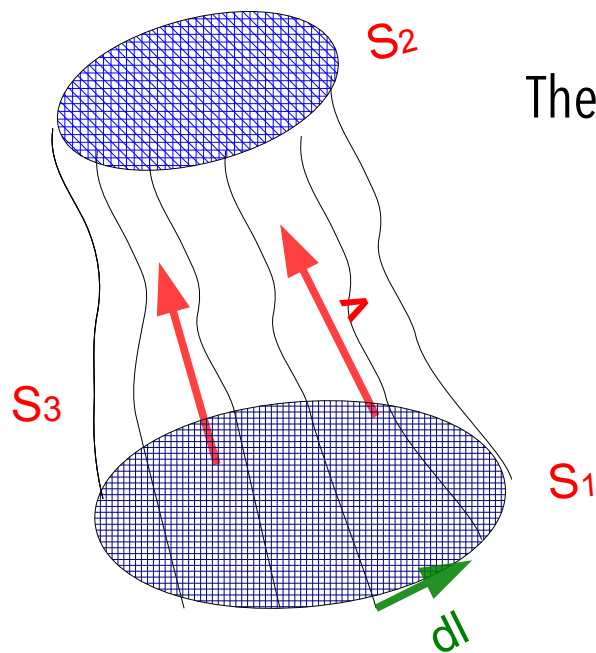
Physical interpretation:

In full analogy to the Kelvin's theory (vortex generation / conservation)

⇒ magnetic field lines cannot be crossed by matter in motion but such matter can stretch and bend the field lines

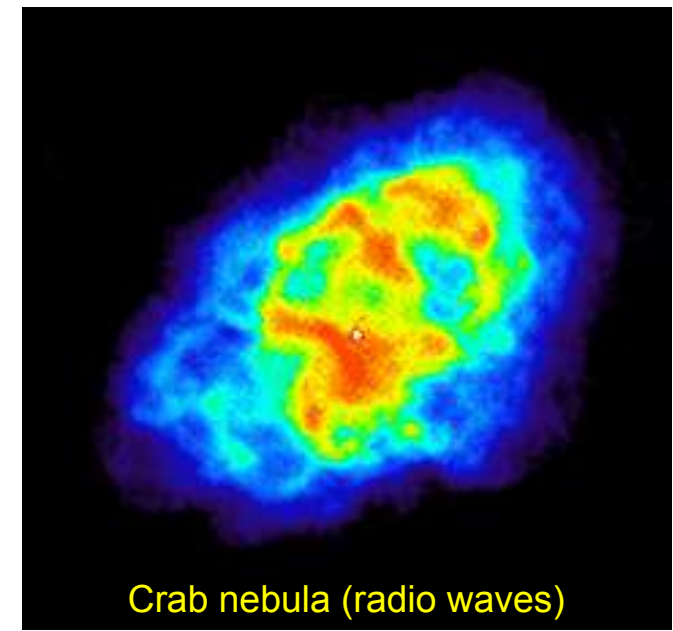
⇒ In case the plasma expands, the field strength decreases and the other way round.

It is therefore possible to create large fields in case of substantial compression of the plasma.



There are many astrophysical examples.

WD, NS, IS & IG plasma, etc.



Crab nebula (radio waves)

Plasma have an additional pressure arising from the H field: depending on their balance with the dynamical contributions, the fate of the plasma will be determined

Magnetic pressure

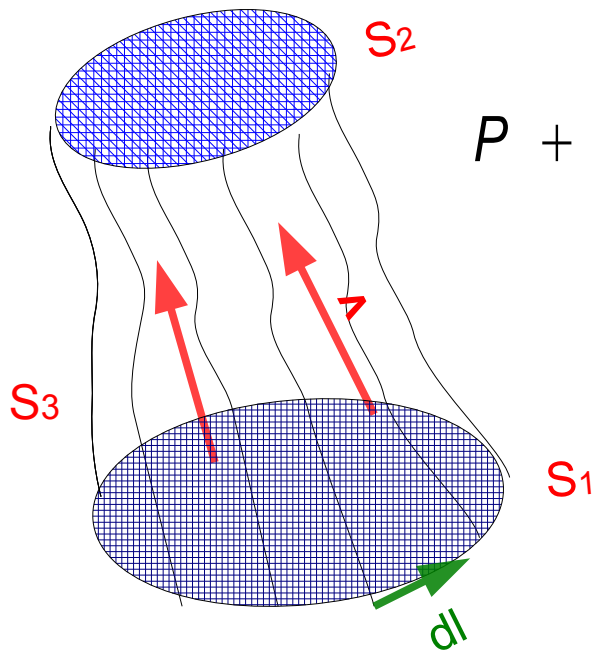
$$P + \rho v^2 < \frac{H^2}{8\pi}$$

field lines will be moderately bent :

the field will confine the plasma

$$P + \rho v^2 \gg \frac{H^2}{8\pi}$$

matter motion will sweep magnetic field lines





Let's define the magnetic energy as

$$W_H = \iiint \frac{H^2}{8\pi} dV = \int_V \frac{H^2}{8\pi} dV$$

Let's consider the time variation of the magnetic energy

$$\begin{aligned} \frac{\partial W_H}{\partial t} &= \frac{1}{4\pi} \int_V \left(\vec{H} \cdot \frac{\partial \vec{H}}{\partial t} \right) dV = \frac{1}{4\pi} \int_V \vec{H} \cdot \left(\vec{\nabla} \times (\vec{v} \times \vec{H}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{H} \right) dV = \\ &= \frac{1}{4\pi} \left(\int_V \vec{H} \cdot \vec{\nabla} \times (\vec{v} \times \vec{H}) dV + \int_V \frac{c^2}{4\pi\sigma} \vec{H} \cdot \nabla^2 \vec{H} dV \right) \end{aligned}$$

Using the vector identities below, skipping a lot (!) of algebra [Details of the computations are in Fanti & Fanti 8.30]

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{H}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{H}) - \nabla^2 \vec{H} \qquad \vec{a} \cdot (\vec{\nabla} \cdot \vec{b}) = \vec{\nabla} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot \vec{\nabla} \times \vec{a}$$

$$\begin{aligned} \frac{\partial W_H}{\partial t} &= \frac{1}{4\pi} \int_V \left(\nabla \cdot [(\vec{v} \times \vec{H}) \times \vec{H}] + (\vec{v} \times \vec{H}) \cdot \nabla \times \vec{H} \right) dV = \\ &= -\frac{1}{c} \int_V \vec{v} \cdot (\vec{j} \times \vec{H}) dV \qquad \left(\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \vec{H} \right) \end{aligned}$$

FYI,
not for study



Therefore we have:

$$\frac{\partial W_H}{\partial t} = -\frac{1}{c} \int_V \vec{v} \cdot (\vec{j} \times \vec{H}) dV \quad \left(\vec{j} = \frac{c}{4\pi} \vec{\nabla} \times \vec{H} \right)$$

which measures the variation of magnetic energy in a time unit as a consequence of the work done on a unit volume by the **magnetic force**

$$\vec{F}_{\text{mag}} \approx \vec{j} \times \vec{H}$$

$$[\quad] = \rho_e \vec{E} + \vec{j} \times \vec{H}$$

- ☐ Field amplification via magnetic dynamo takes place when kinetic (mechanic) energy is converted into magnetic energy



Dynamo effect:

Turbulent motions ($\vec{v} \neq 0$) in a conductive fluid ($\sigma \rightarrow 0$) can spontaneously generate H fields

$$\frac{\partial \vec{H}}{\partial t} = \nabla \times (\vec{v} \times \vec{H}) + \frac{c^2}{4\pi\sigma} \nabla^2 \vec{H}$$

Turbulent motions imply turbulent H field structure, and likewise vorticity, $\vec{\Omega}$, also the H field can be amplified/damped in a very similar way

$$\vec{H} = \text{const} \cdot \vec{\Omega}$$

If W_H grows, then the field is amplified, unless the energy is dissipated (e.g. Joule effect)

Amplification is quenched when

$$\frac{H^2}{8\pi} \simeq \frac{1}{2} \rho v^2$$

i.e. when the magnetic pressure becomes strong enough to slow/halt turbulence

Examples are: convective regions in the Sun, turbulent intergalactic medium in galaxy clusters, etc.

Magnetic forces: let's interpret what the magnetic force is:

$$\vec{F}_H = \frac{\vec{j} \times \vec{H}}{\rho c} = \frac{1}{4\pi\rho} (\nabla \times \vec{H}) \times \vec{H}$$

using the vector identity $(\nabla \times \vec{H}) \times \vec{H} = (\vec{H} \cdot \vec{\nabla}) \vec{H} - \frac{1}{2} \vec{\nabla} H^2$

This operator represents H times the derivative along the field line

$$\vec{H} \cdot \vec{\nabla} = H \frac{\partial}{\partial s}$$

It can be interpreted as a resistive term to variations/changes of the local magnetic field geometry

$$\vec{F}_H = \frac{1}{\rho} \frac{1}{4\pi} (\vec{H} \cdot \vec{\nabla}) \vec{H} - \frac{1}{\rho} \vec{\nabla} \left(\frac{H^2}{8\pi} \right)$$

tension along
field lines

isotropic H
pressure



$$\frac{1}{4\pi} (\vec{H} \cdot \vec{\nabla}) \vec{H} = \frac{1}{4\pi} \vec{\nabla} (\vec{H} \vec{H})$$

stress tensor

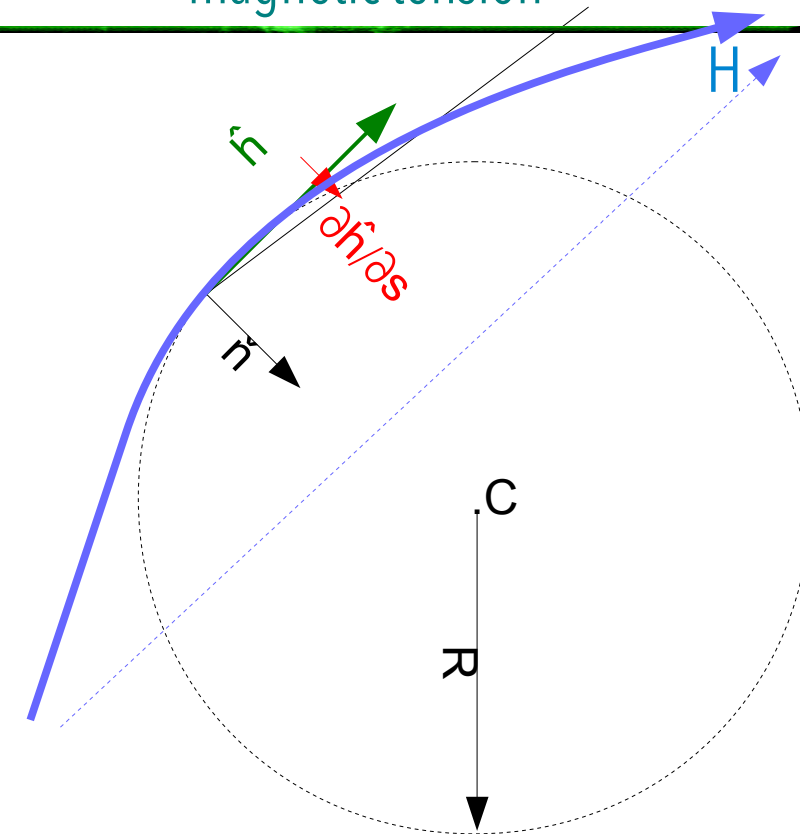
$$= \frac{1}{4\pi} H \frac{\partial}{\partial s} H \hat{h} = \frac{H^2}{4\pi} \frac{\partial \hat{h}}{\partial s} + \frac{\hat{h}}{8\pi} \frac{\partial H^2}{\partial s}$$

1. perpendicular to local H ($\hat{h}$) direction

If R is the curvature radius of a field line it can be written as $H^2/4\pi R$. This term is 0 for a uniform H

2. along local H ($\hat{h}$) direction

It is significant **only** in case of substantial variation of H field intensity; identical to $[1/\rho] \nabla (\mu/8\pi)$ but with opposite sign (**but computed along the field line**). This term is 0 for a uniform H



Deflected *magnetic field lines* may therefore produce two forces *perpendicular to the fluid velocity* and *to the local direction of the H field itself*:

1. directed towards C, aims to shorten the field line (like a guitar string!)
2. opposing to gradient of the magnetic pressure along field lines

FYI,
Computations
never asked



$$\vec{F}_H = \frac{1}{\rho} \frac{1}{4\pi} (\vec{H} \cdot \vec{\nabla}) \vec{H} - \frac{1}{\rho} \vec{\nabla} \left(\frac{H^2}{8\pi} \right)$$

tension along
field lines

isotropic H
pressure

These two terms produce an anisotropic mechanical effect, depending on H geometry

Limiting cases:

1. Uniform H field, tension is zero

2. Disordered H field on scales much smaller than the plasma length (volume):

*Both terms play a role and in general we end out with an **additional** isotropic pressure (a lot of algebra is Skipped!)*

$$\frac{1}{3} \frac{\langle H^2 \rangle}{8\pi}$$

[examples: compression Δl along H and perpendicularly to H]

FYI,
Computations
never asked

Waves in a magnetic field:

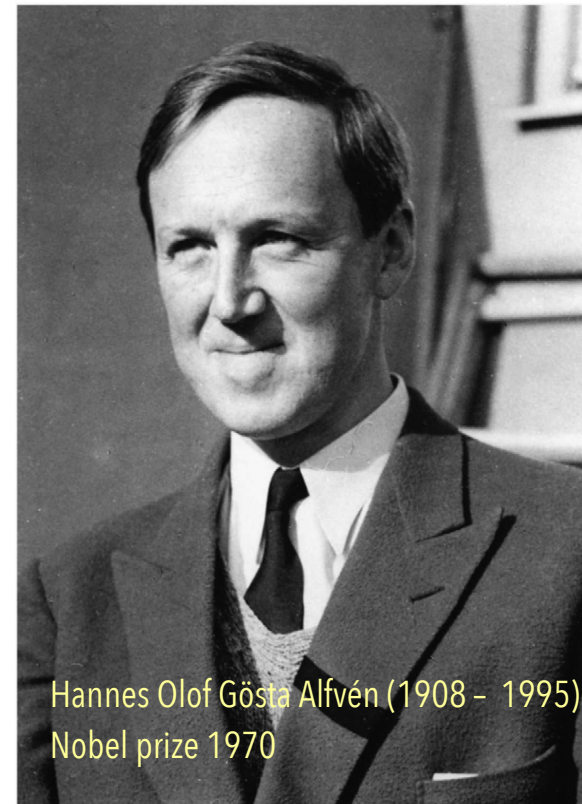
also when $\rho = \text{const}$ (incompressible) and $\eta = 0$ there could be transverse waves as a consequence of $\frac{H^2}{4\pi}$ allowing this effect since this term represents an oscillator.

In general if we consider a small perturbation (with the addition of $B = B_{0z}$) the final solution must satisfy

$$(\omega^2 - v_a^2 k_{\parallel}^2) [\omega^4 - \omega^2 (c_s^2 + v_a^2) k^2 + c_s^2 v_a^2 k^2 k_{\parallel}^2] = 0$$

These are known as **Alfvén waves** (MHD waves) which propagate **along the field lines** with a typical velocity v_A which can be defined as follows:

$$v_A = \sqrt{\frac{H^2}{4\pi\rho}} = \frac{H}{\sqrt{4\pi\rho}}$$



Hannes Olof Gösta Alfvén (1908 – 1995)
Nobel prize 1970



Therefore Alfvén waves are the **only** way for propagating perturbations in an incompressible, non viscous, magnetized plasma.

In case the fluid is also compressible there are also longitudinal waves.

The general case is

$$v^2 = \frac{c_s^2 + v_a^2}{2} \left[1 \pm \sqrt{1 - 4 \cos^2 \theta \frac{c_s^2 v_a^2}{(c_s^2 + v_a^2)^2}} \right]$$

1. perpendicularly to local H direction ($\theta = 0$) we have sonic waves propagating with c_s
2. along field lines we have **magneto-sonic** waves propagating with

$$c_{ms} = \sqrt{c_s^2 + v_A^2}$$

In presence of a shock and assuming that there is a H field parallel to the shock surface, the momentum conservation becomes:

$$\rho_1 v_1'^2 + p_1 + \frac{H_1^2}{8\pi} = \rho_2 v_2'^2 + p_2 + \frac{H_2^2}{8\pi}$$

which gives, (after some algebra)

$$\frac{H_2}{H_1} = \frac{\rho_2}{\rho_1}$$

therefore in case of an adiabatic shock ($c_s^2 = \Gamma(p/\rho)$)

the H field can be compressed, i.e. amplified, by a factor of 4.

Indeed, the presence of a magnetic field reduces the maximum compression rate from 4 to 3 (limiting case).

[See Fanti & Fanti, 8.4.7]

In case of isothermal shock $T_2 = T_1, c_s^2 = (p/\rho)$ then

$$p_1 = \rho_1 c_{s1}^2 ; p_2 = \rho_2 c_{s2}^2 \quad \text{with} \quad \rho v^2 + p = \text{const}$$

and the eq. of momentum conservation

$$\rho_1 v_1^2 + \rho_1 c_{s1}^2 + \frac{H_1}{8\pi} = \rho_2 v_2^2 + \rho_2 c_{s2}^2 + \frac{H_2}{8\pi} = \rho_2 v_2^2 + \rho_2 c_{s2}^2 + \left(\frac{\rho_2}{\rho_1}\right)^2 \frac{H_1}{8\pi}$$

and reintroducing the Alfvén speed, in case of $v_A \gg c_{s1}, c_{s2}$ and $\rho_2 > \rho_1$

$$\frac{\rho_2}{\rho_1} = \sqrt{2} \frac{v_{SH1}}{v_{A1}} = K$$

where K is known as **Magnetic Mach number (Karman)**

FYI,
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never asked



⇒ HD and MHD equations differ given the presence of the H field. ⇐

Mass conservation is not affected, momentum & energy conservation need to be modified.

For example, the magnetic energy must be entered into the energy conservation equation.

Let's consider the eq. of momentum conservation where the magnetic force has to be added

e.g. Description of a magnetized fluid motion:

Navier-Stokes' equation in case of a H field

[we consider internal dynamic but not magnetic viscosity...] i.e. $\frac{\eta_m}{\rho} \nabla^2 \vec{H} \simeq 0$

$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v} \cdot \vec{\nabla}) \vec{v} - \frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\eta}{\rho} \nabla^2 \vec{v} + \frac{\vec{j} \times \vec{H}}{\rho c}$$



$$\frac{\partial \vec{v}}{\partial t} = -(\vec{v} \cdot \vec{\nabla}) \vec{v} - \frac{1}{\rho} \vec{\nabla} p - \vec{\nabla} \phi + \frac{\eta}{\rho} \nabla^2 \vec{v} + \frac{\vec{j} \times \vec{H}}{\rho c}$$

taking a linear approximation of this equation:

$$\begin{aligned} \frac{\partial v}{\partial t} &\simeq - \frac{v^2}{L} - \frac{p}{\rho L} - g + \frac{\eta}{\rho} \frac{v}{L^2} + \frac{H^2}{4\pi\rho L} \\ &= \frac{v^2}{L} \left(-1 - \frac{p}{\rho v^2} - \frac{gL}{v^2} + \frac{\eta}{\rho v L} + \frac{H^2}{4\pi\rho v^2} \right) \\ &= \frac{v^2}{L} \left(-1 - \frac{1}{M^2} - \frac{1}{F} + \frac{1}{R} + \frac{1}{K^2} \right) \end{aligned}$$

$$R = \frac{\rho v L}{\eta} \quad \text{Reynolds}$$

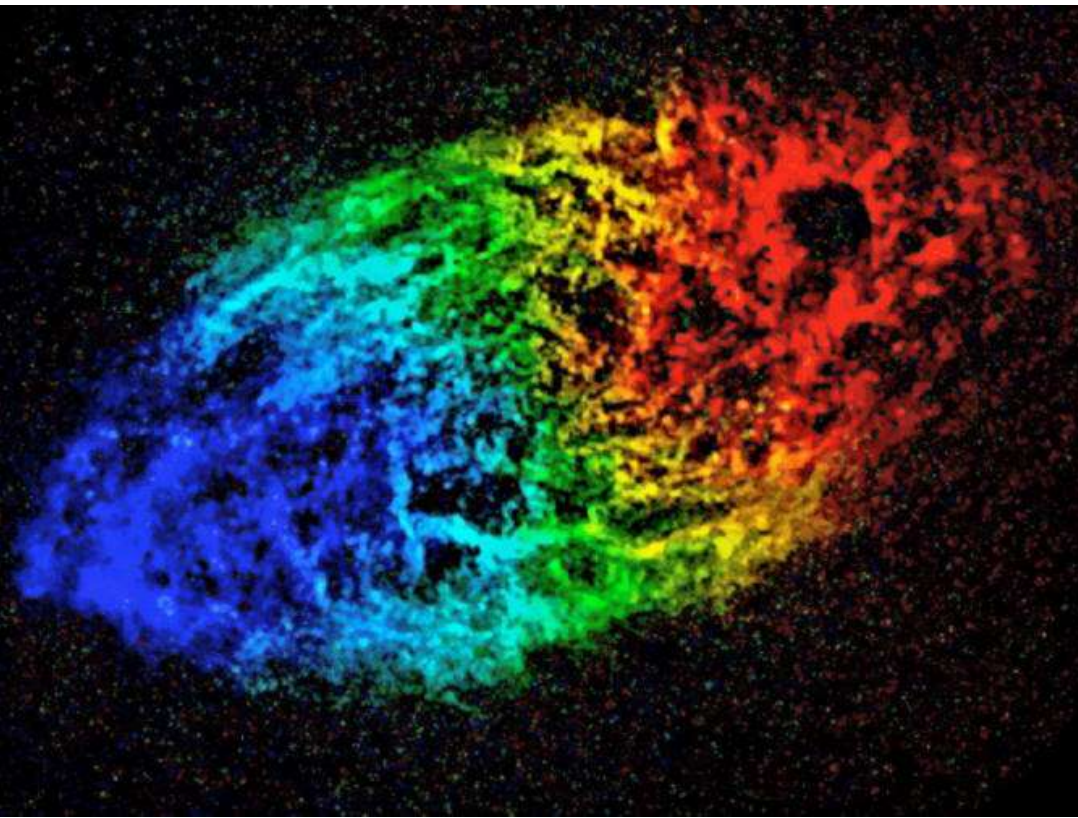
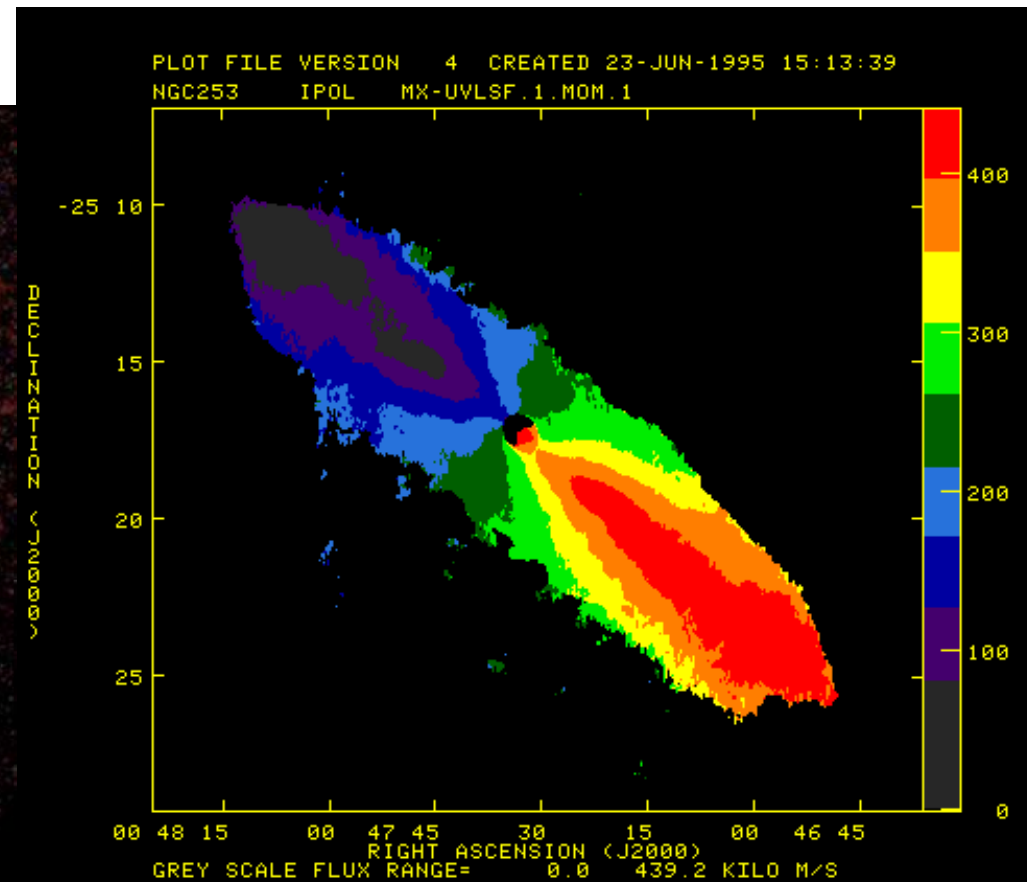
$$K = \frac{v}{v_a} = \frac{v}{\sqrt{H^2/4\pi\rho}} \quad \text{Karman}$$

$$M = \frac{v}{c_s} = \frac{v}{\sqrt{p/\rho}} \quad \text{Mach}$$

$$F = \frac{v^2}{gL} \quad \text{Froude}$$

⇒ MHD systems with the same combination of M, F, R, K have the same behaviour; self – similar motions – astrophysical conditions can be reproduced in lab if they have the same numerical set

- a *stable* (steady state, i. e. $\partial / \partial t = 0$) *configuration* can be interesting in astrophysics since it is more likely to be found than *unstable configurations*. It can be considered "persistent".
- *perturbations are either diminished or there is the possibility of oscillations or waves about the equilibrium.*





Instabilities in many different forms in (M-)-hydrodynamical systems

- *equilibrium configuration* which undergoes to a small perturbation studied in its linear approximation.
- the *perturbation is described by plane waves* $e^{i(\omega t - \vec{k} \cdot \vec{x})}$
- *dispersion relation* is obtained $\omega(k)$
- *imaginary frequencies pinpoint the onset of instabilities*

A fluid in motion relative to another body (fluid) may alter its own configuration if small perturbations grow rather than damping out

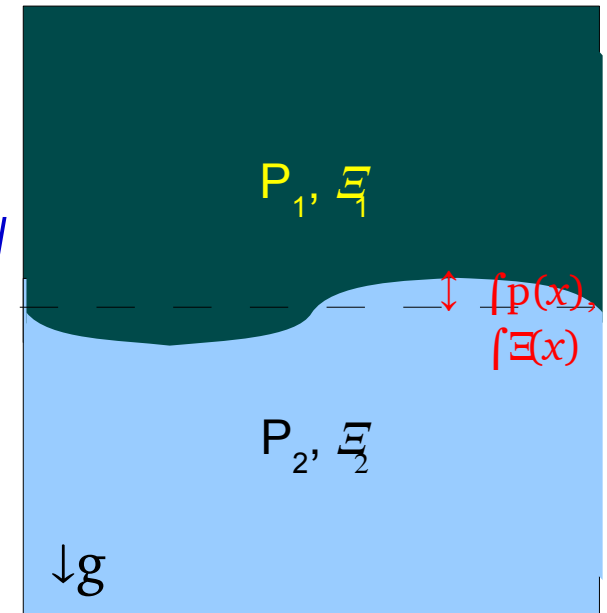
common astrophysical instabilities:

- | | | |
|--------------------------|---------------|------------------------------------|
| <i>Rayleigh-Taylor</i> | $\Rightarrow$ | <i>“mushroom” clouds</i> |
| <i>Kelvin-Helmholtz</i> | $\Rightarrow$ | <i>shear flow roll-up</i> |
| <i>Richtmyer-Meshkov</i> | $\Rightarrow$ | <i>shock interface instability</i> |
| <i>Convective</i> | $\Rightarrow$ | <i>buoyant overturn</i> |
| <i>Thermal</i> | $\Rightarrow$ | <i>runaway cooling</i> |
| <i>Jeans</i> | $\Rightarrow$ | <i>gravitational collapse</i> |

Rayleigh-Taylor instability

Two incompressible, not mixable, fluids develop with a sinusoidal perturbation at the interface:

- fluids at rest; $\rho_1 > \rho_2$; constant (in time) gravitational field*
- unstable configuration in absence of surface tension*
- bubbles (fingers) of less dens fluid rise to compensate bubbles (fingers) of heavier fluid dripping down*



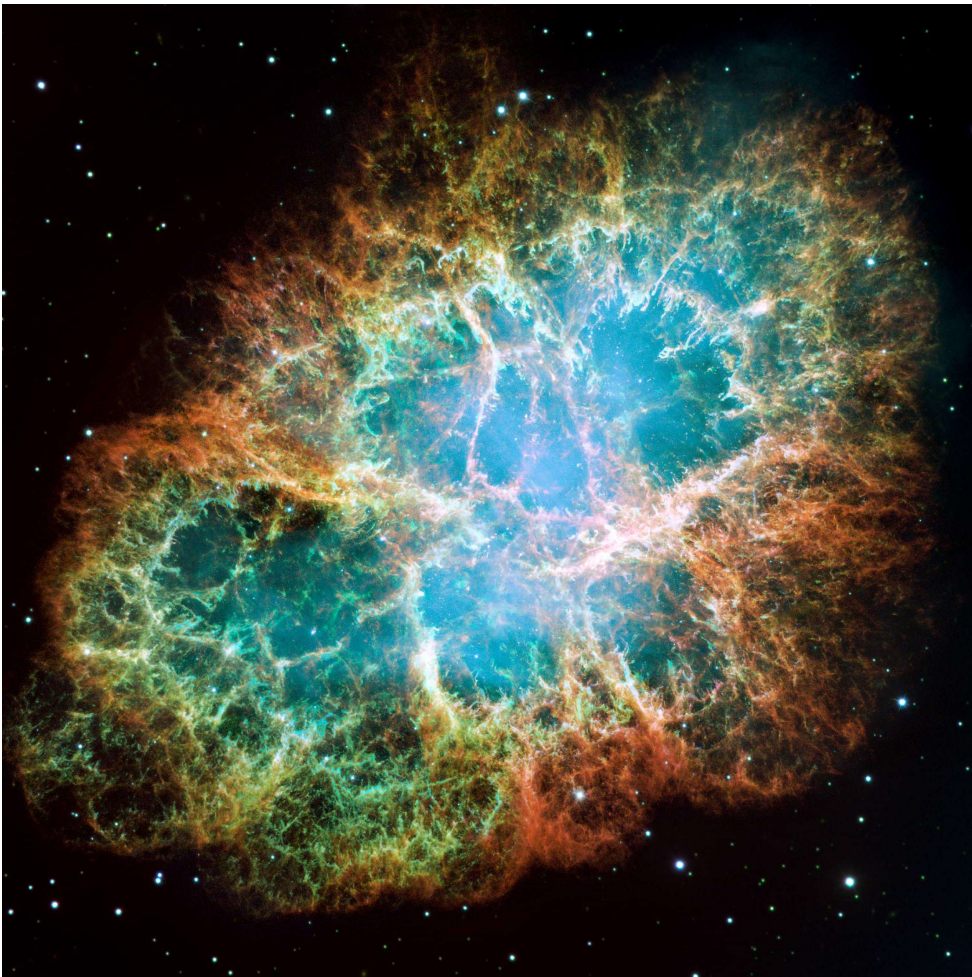
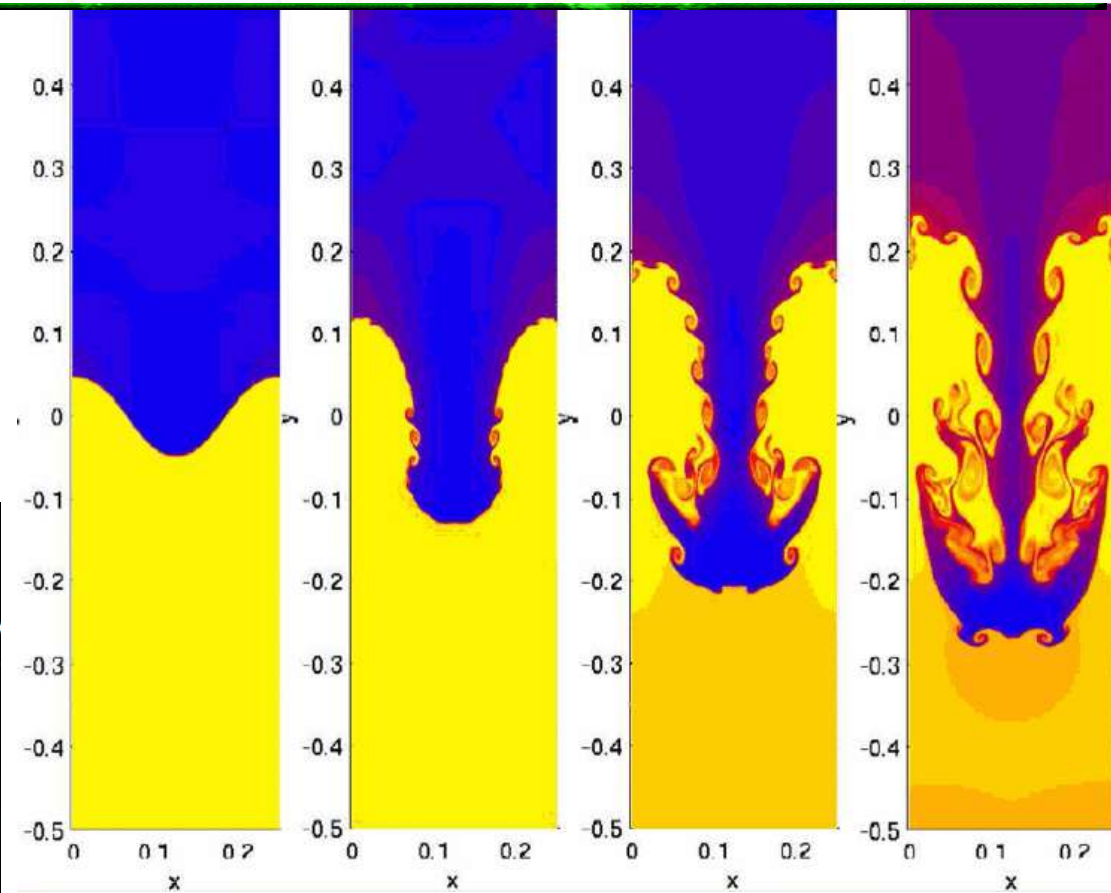
$$\omega^2 = k^2 g z_1^* \frac{\rho_2 - \rho_1}{\rho_2} \quad \text{is negative if } \rho_2 > \rho_1$$

$$\left| \frac{dT}{dz} \right| < \left(1 - \frac{1}{\Gamma} \right) \frac{T}{p} \left| \frac{dp}{dz} \right|$$

Schwarzschild stability criterion

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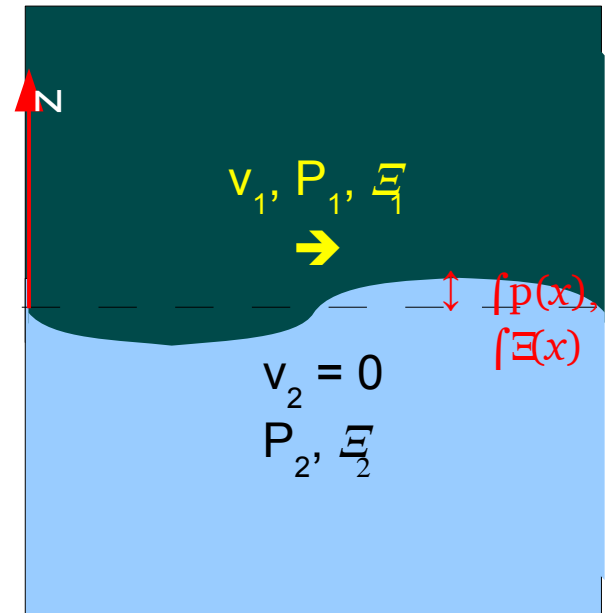
Rayleigh-Taylor instability



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Kelvin-Helmoltz instability

Two incompressible, immiscible, fluids in relative motion at a speed v , with a sinusoidal perturbation at the interface:



– Linearized Euler equation becomes

$$\rightarrow \frac{\partial \vec{v}}{\partial t} = -\frac{\nabla p}{\rho} \quad \text{then take the divergence}$$

$$\nabla^2 p = 0 \quad \rightarrow \quad \nabla \cdot \vec{v} = 0$$

$$\rightarrow \delta p = f(z) e^{[i(kx - \omega t)]} \quad \text{leads to} \quad \frac{\partial^2 f(z)}{\partial z^2} - k^2 f(z) = 0$$

solutions: $f(z) \sim e^{\pm kz}$ $f(z) \sim e^{kz}$ ruled out, (it diverges at large z)

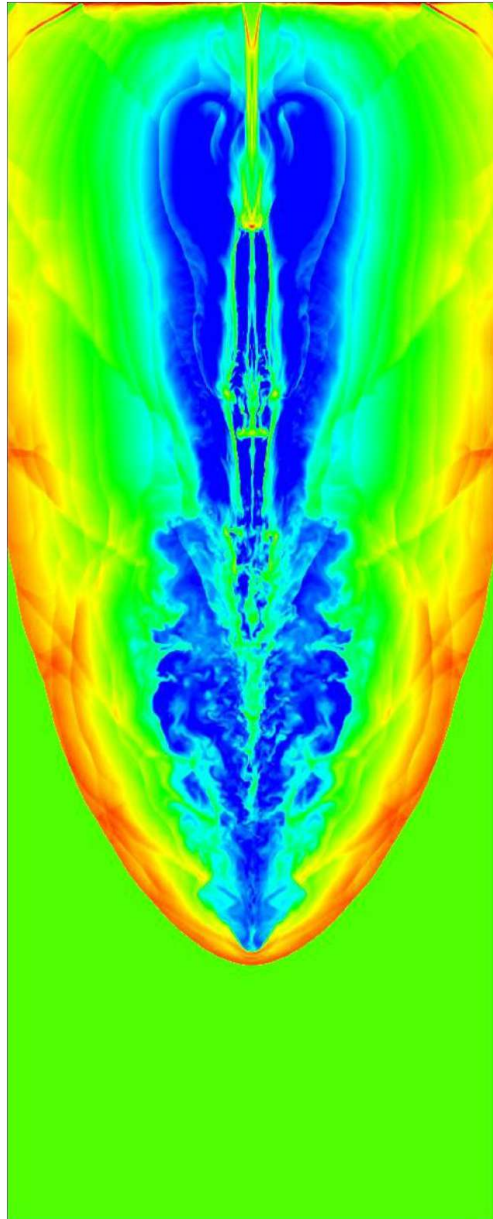
above the boundary surface: $\delta P_1 \sim e^{kz} e^{[i(kx - \omega t)]}$

$$\frac{\partial \delta v_z}{\partial t} + v \frac{\partial \delta v_z}{\partial x} = -\frac{k \delta P_2}{\rho_2}$$

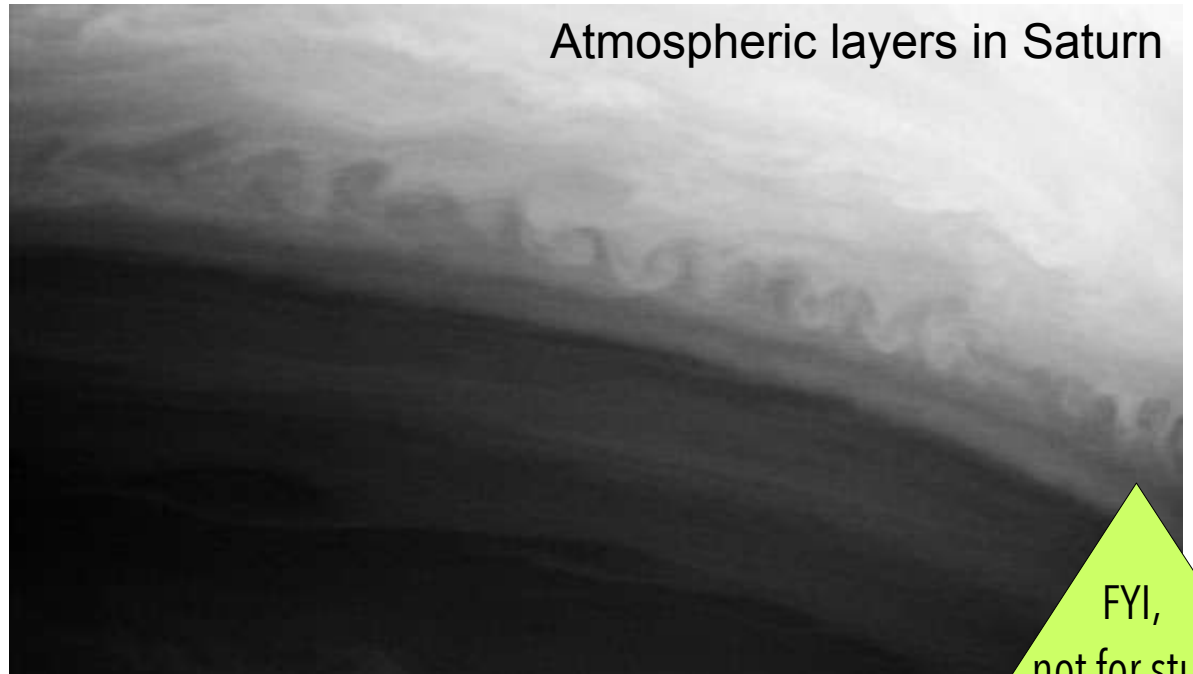
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clouds over mt. Duval (Australia)



Atmospheric layers in Saturn



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➤ **What is a fluid (plasma)**

- Fluid Equations (conservations of mass, momentum, energy)
- Propagation of perturbations, sound speed (Alfven speed)
- Supersonic perturbations, conversion of kinetic energy into something else
- No (stable) E field in the Universe
- Widespread H fields (generally weak, but also very strong)
- Energy in H field may be relevant
- (Often, fluid instabilities are responsible of interesting astrophysical phenomena)