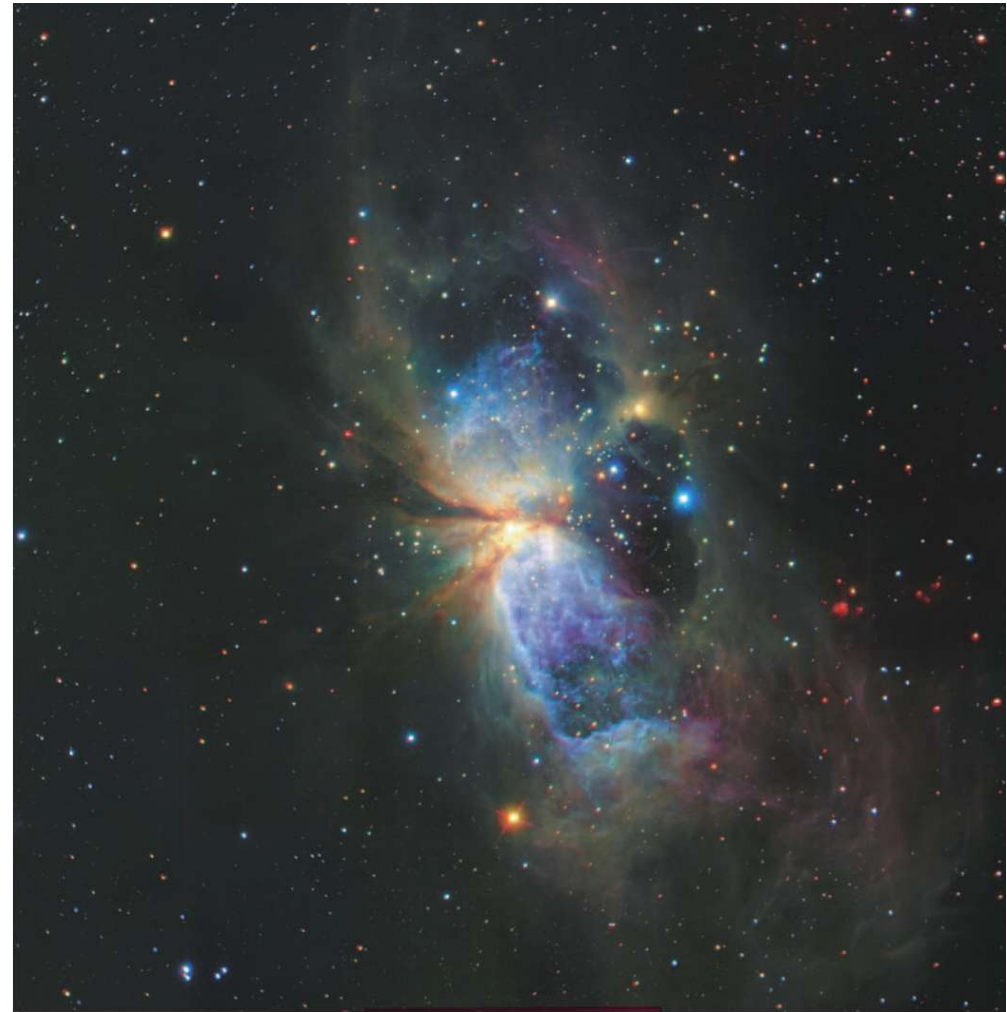


Radiative Processes and more

- A solid background in Modern Physics is needed
- Summary of basic concepts from our Bachelor in Astronomy. Demonstrations will be dropped.
- A number of key-words will be associated to diagrams, diagnostics implying given physical conditions, etc
-



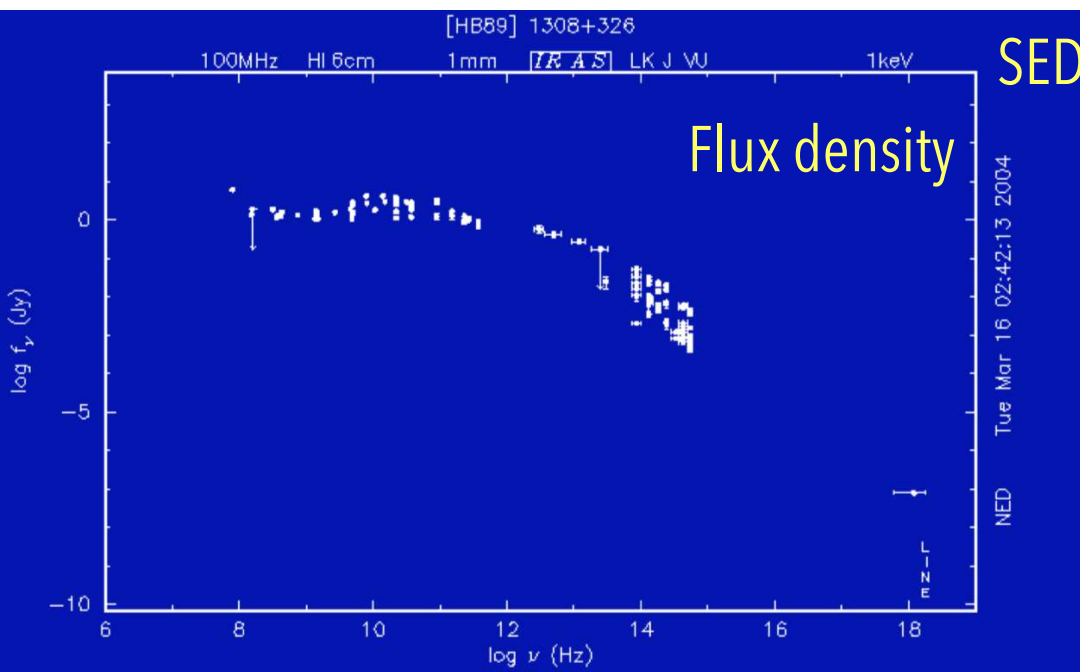
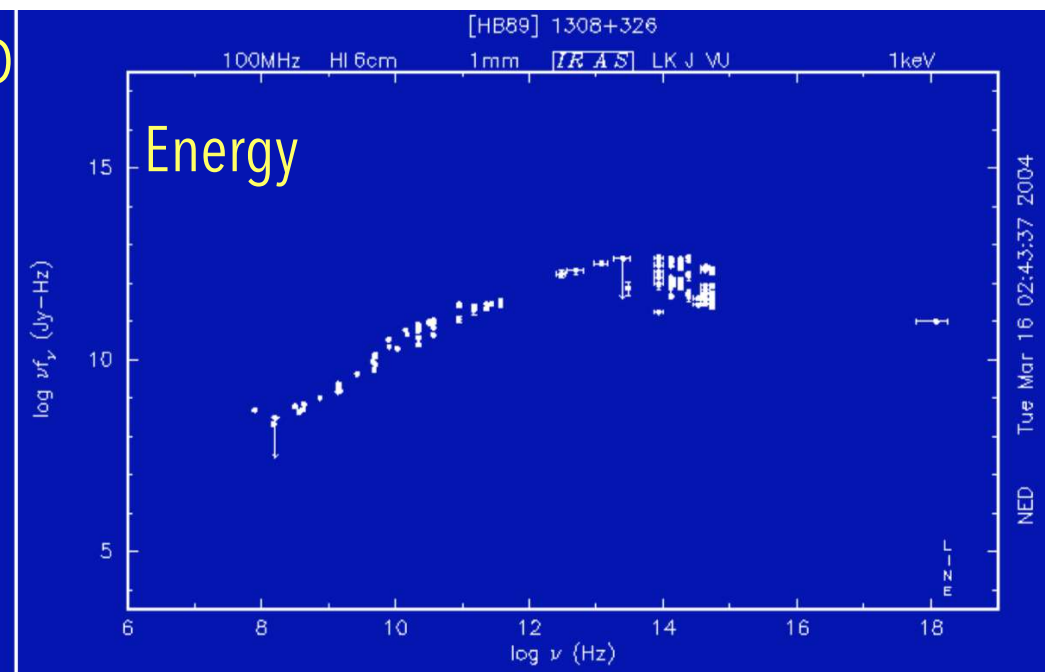
—False, three-color composite image of S106 in the Subaru CISCO observations (J band [blue], H band [green], K 0 band [red]). The field of view of the mosaic is approximately $50'' \times 50''$. North is up, and east is to the left.



Keywords (at the moment w/o meaning)

- Emission Spectrum
- Radiative Transfer, Black Body radiation
- Larmor's formula and the acceleration of an electric charge
- Bremsstrahlung
- Synchrotron
- Scattering processes (Thomson, Compton, Inverse Compton)
- Line emission

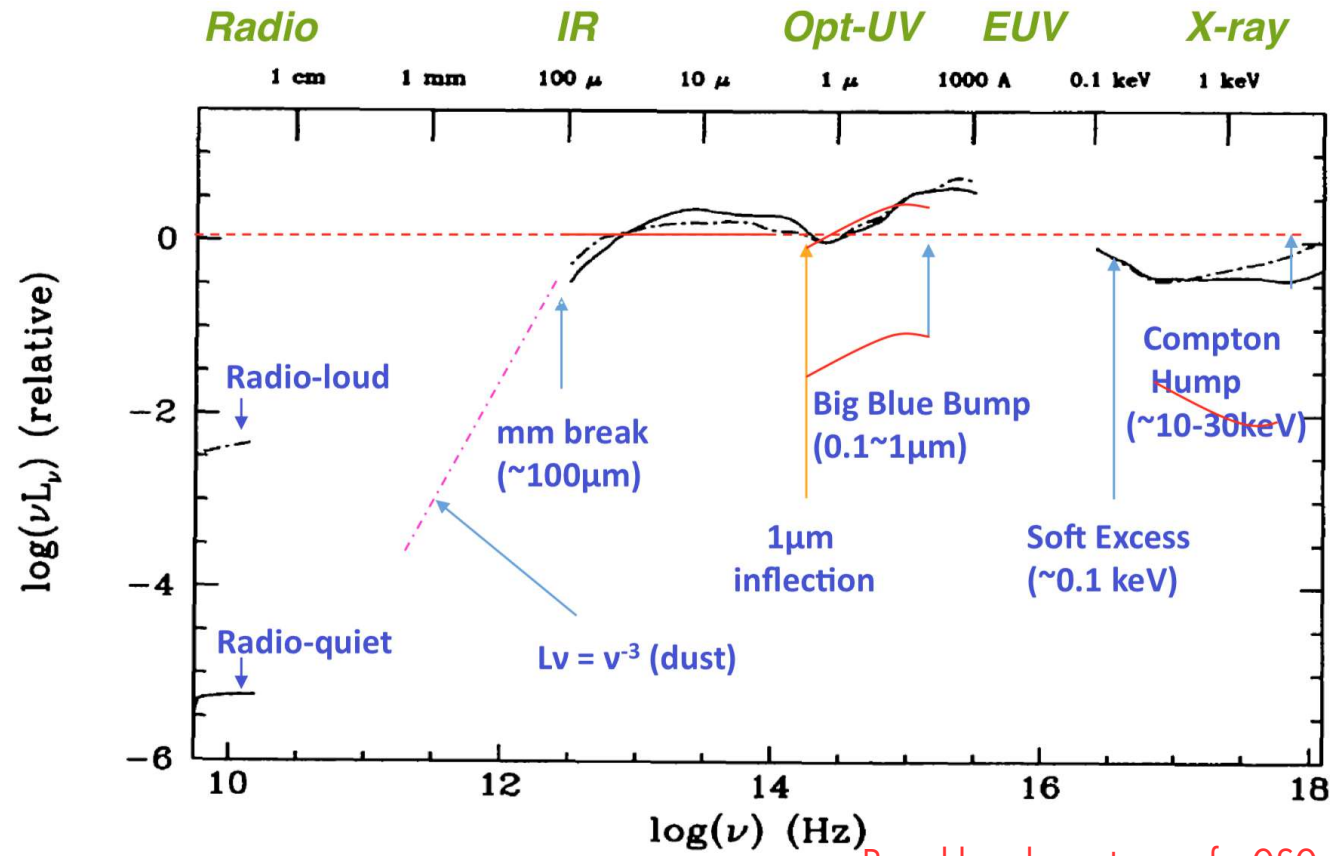
Emission Spectrum



- **Broad**/narrow band
- Number of photons (emitted/received) as a function of frequency (energy) [LEFT]
- Energy (emitted/received) as a function of frequency (energy) [RIGHT]

A typical expression can be: $\mathbf{F}(\nu) = \text{something}$ depending on the origin of the radiation

Emission Spectrum



Broad band spectrum of a QSO

Elvis et al., 1994, ApJS, 95, 1

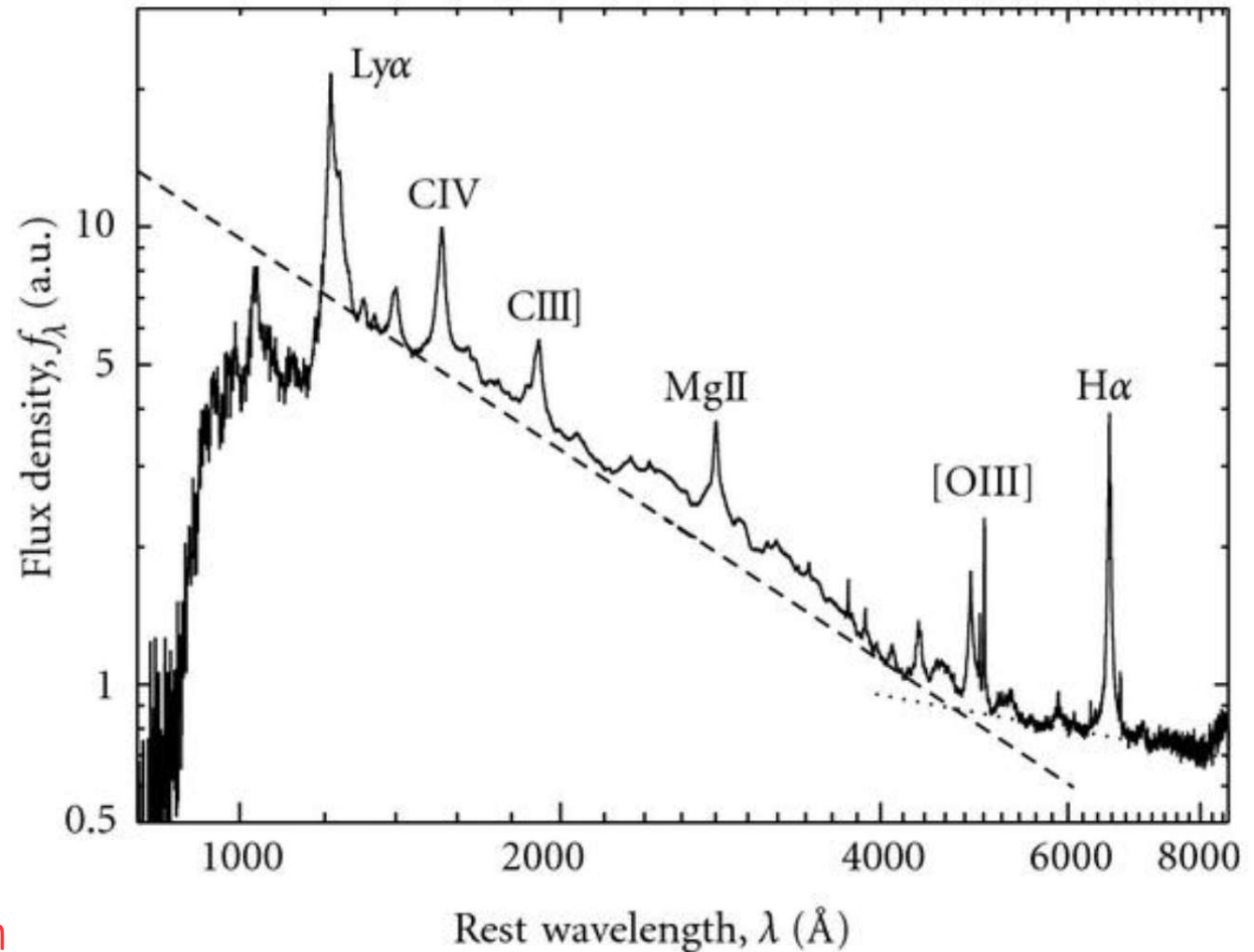
- **Broad**/narrow band
- Energy (emitted/received) as a function of frequency (energy)

Emission Spectrum

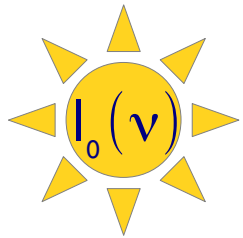
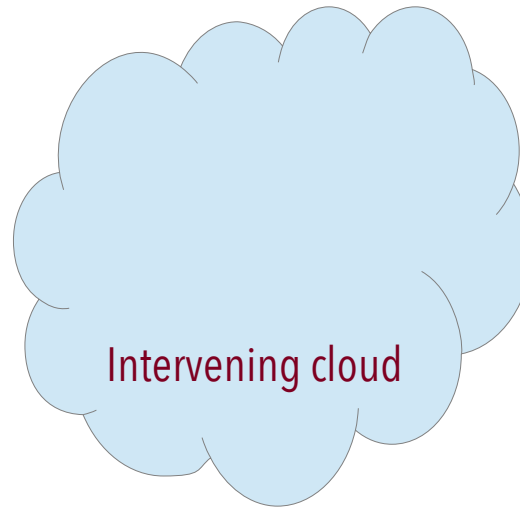
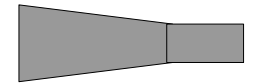
- Number of photons (emitted/received) as a function of frequency (energy)

- Broad/**narrow** band

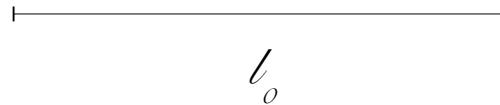
- UV-Optical spectrum of a synthetic "average" QSO $\Rightarrow$



- Note: Continuum + line emission



observer



$$I(v) = I_0(v)e^{-\tau(v)} + S(v)[1 - e^{-\tau(v)}]$$

$$S(v) = \frac{j(v)}{\mu(v)} \text{ is the } \mathbf{source\ function}$$

$$\tau(v) = \mu(v)l_o \quad \mathbf{opacity}$$

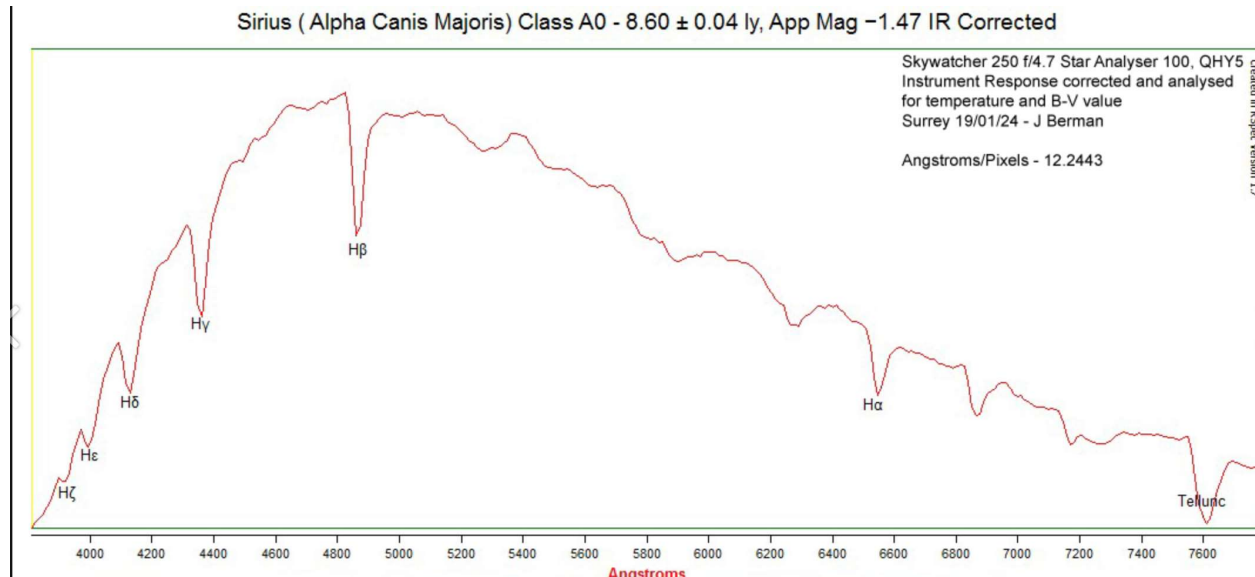
What does happen when the cloud is optically thick?

What does happen when the cloud is optically thin?

STRONGLY dependent on frequency (and cloud physics)



Radiative Transfer & Black Body radiation



example(s) $\uparrow \rightarrow$

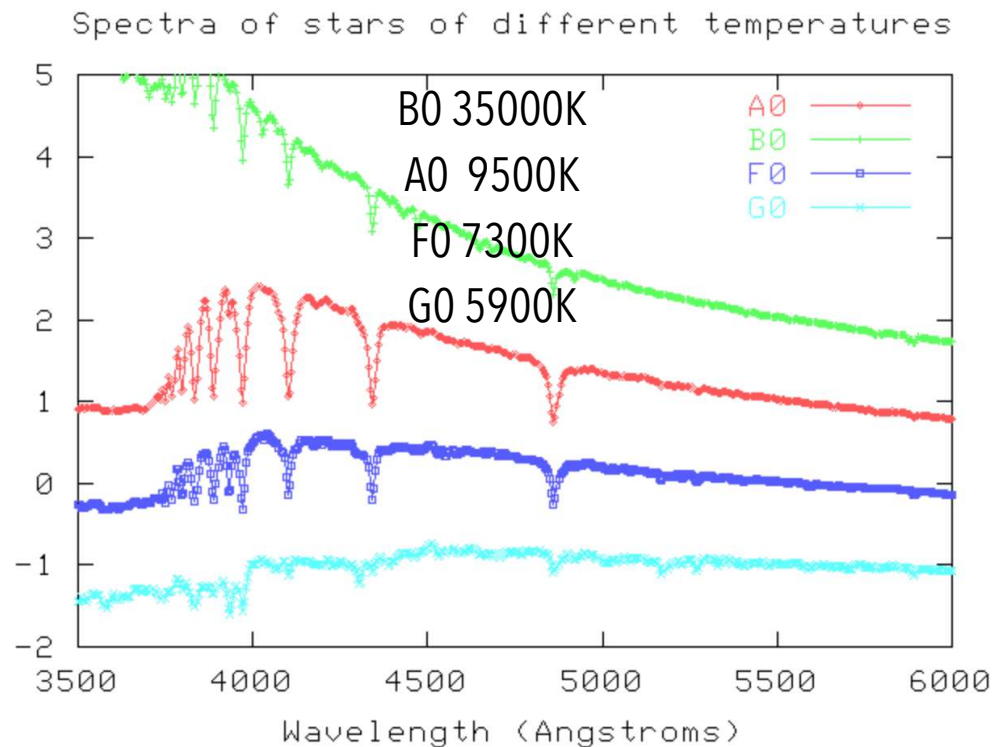
$$I(\nu) = I_0(\nu)e^{-\tau(\nu)} + S(\nu)[1 - e^{-\tau(\nu)}]$$

What does happen when the cloud is optically thick?

What does happen when the cloud is optically thin?

$\rightarrow$ STRONGLY dependent on frequency
(and cloud physics)

What is the limit if $\tau \rightarrow \infty$



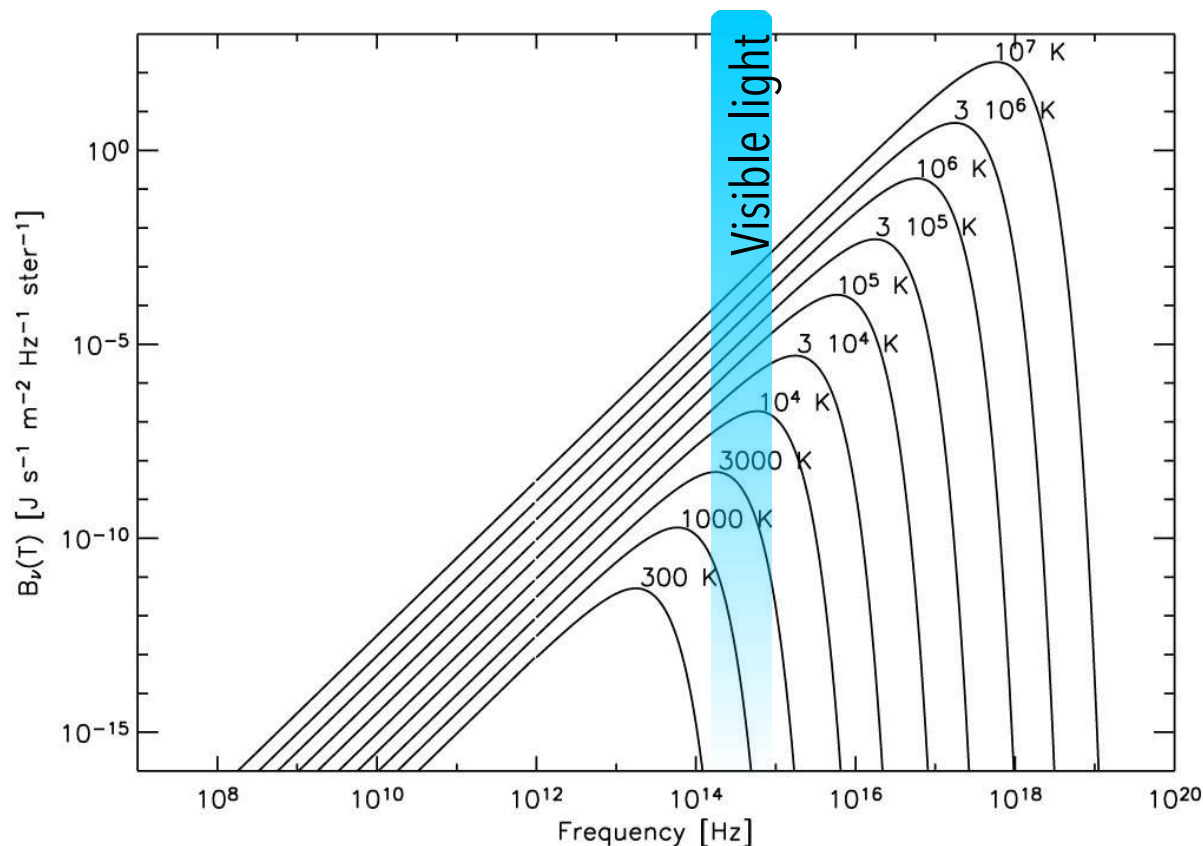
Definition:

BLACK BODY = an idealized **opaque**, non-reflective body like an empty cavity in **thermal equilibrium**, isolated, where only radiation is present. Given its definition, in principle, a BB is not visible

Widespread in Astrophysics, with excellent approximations.

The spectral distribution of radiation described by the Planck's law

$$B(\nu, T) = 2 \frac{h \nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$



Similar expression as a function of λ

$$\nu = \frac{c}{\lambda} \rightarrow d\nu = -\frac{c}{\lambda^2} d\lambda$$

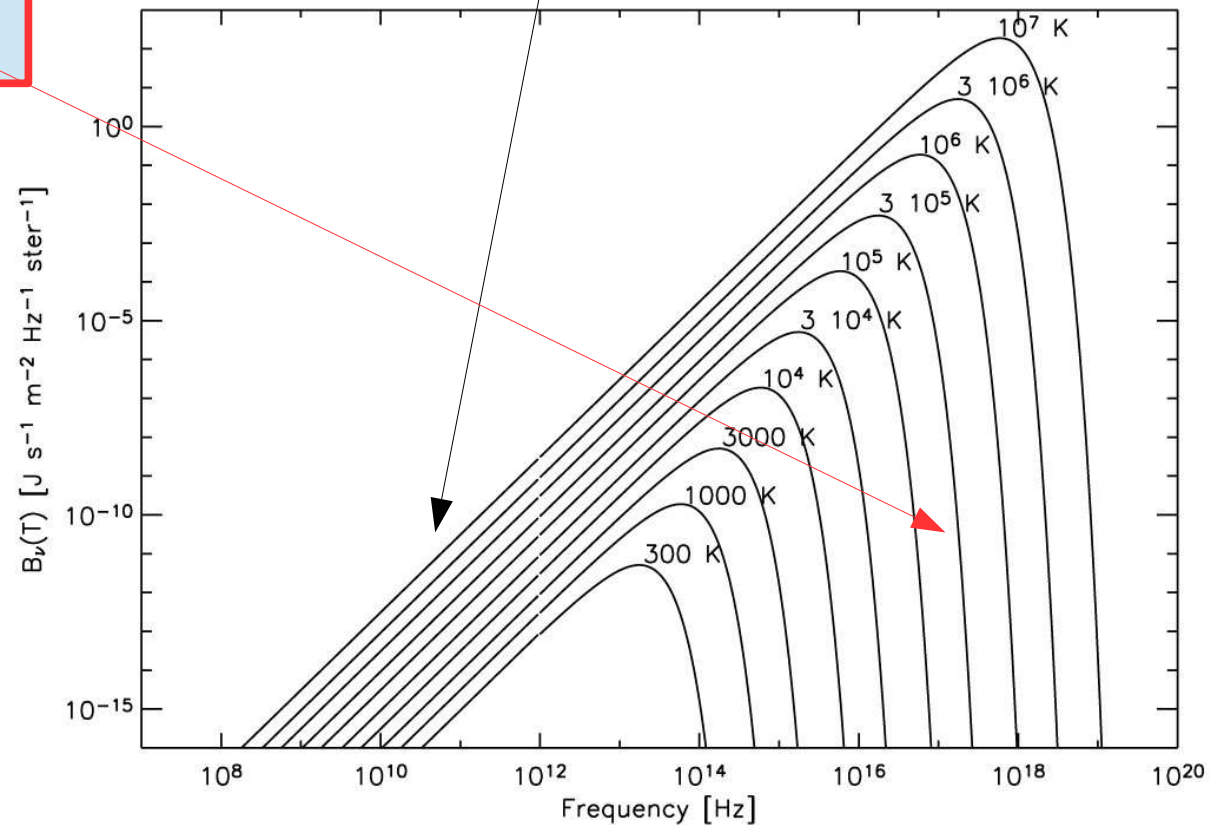
Then the two expressions are

DIFFERENT: $(\lambda_{\max} \cdot \nu_{\max} \neq c)$

Asintotic behaviour $B(\nu, T) = 2 \frac{h \nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$

$\nu \rightarrow 0$ $e^{\frac{h\nu}{kT}} - 1 = 1 + \frac{h\nu}{kT} - 1 = \frac{h\nu}{kT} \Rightarrow B(\nu, T) = 2kT \cdot \left(\frac{\nu}{c}\right)^2$ **Rayleigh — Jeans**

$\nu \rightarrow \infty$ $B(\nu, T) = 2 \frac{h \nu^3}{c^2} \cdot e^{-\frac{h\nu}{kT}}$ **Wien**





Radiative Transfer & Black Body radiation

$$B(\nu, T) = 2 \frac{h \nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$B(\lambda, T) = 2 \frac{hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

Remarkable features:

A given curve is determined by T

No crossing point among different $B(\nu, T)$ or different $B(\lambda, T)$

Given a point in the plot it is possible to determine T



Radiative Transfer & Black Body radiation

$$B(\nu, T) = 2 \frac{h \nu^3}{c^2} \frac{1}{e^{\frac{h\nu}{kT}} - 1}$$

$$B(\lambda, T) = 2 \frac{hc^2}{\lambda^5} \frac{1}{e^{\frac{hc}{\lambda kT}} - 1}$$

Wien displacement law

$$\left(\frac{\partial B(\nu, T)}{\partial T} \right)_{\nu=\nu_{\max}} = 0$$

find the peak frequency: that becomes equivalent to solve . $x = 3(1 - e^{-x})$ given that $x \equiv \frac{h \nu_{\max}}{kT}$

$$\text{approx root } \frac{h \nu_{\max}}{kT} = 2.82 \quad \nu_{\max} = 2.82 \frac{k}{h} T \rightarrow = 5.88 \cdot 10^{10} T \text{ Hz deg}^{-1}$$

and for wavelengths: that becomes equivalent to solve

$$\left(\frac{\partial B(\lambda, T)}{\partial T} \right)_{\lambda=\lambda_{\max}} = 0; y = \frac{hc}{\lambda_{\max} kT}$$

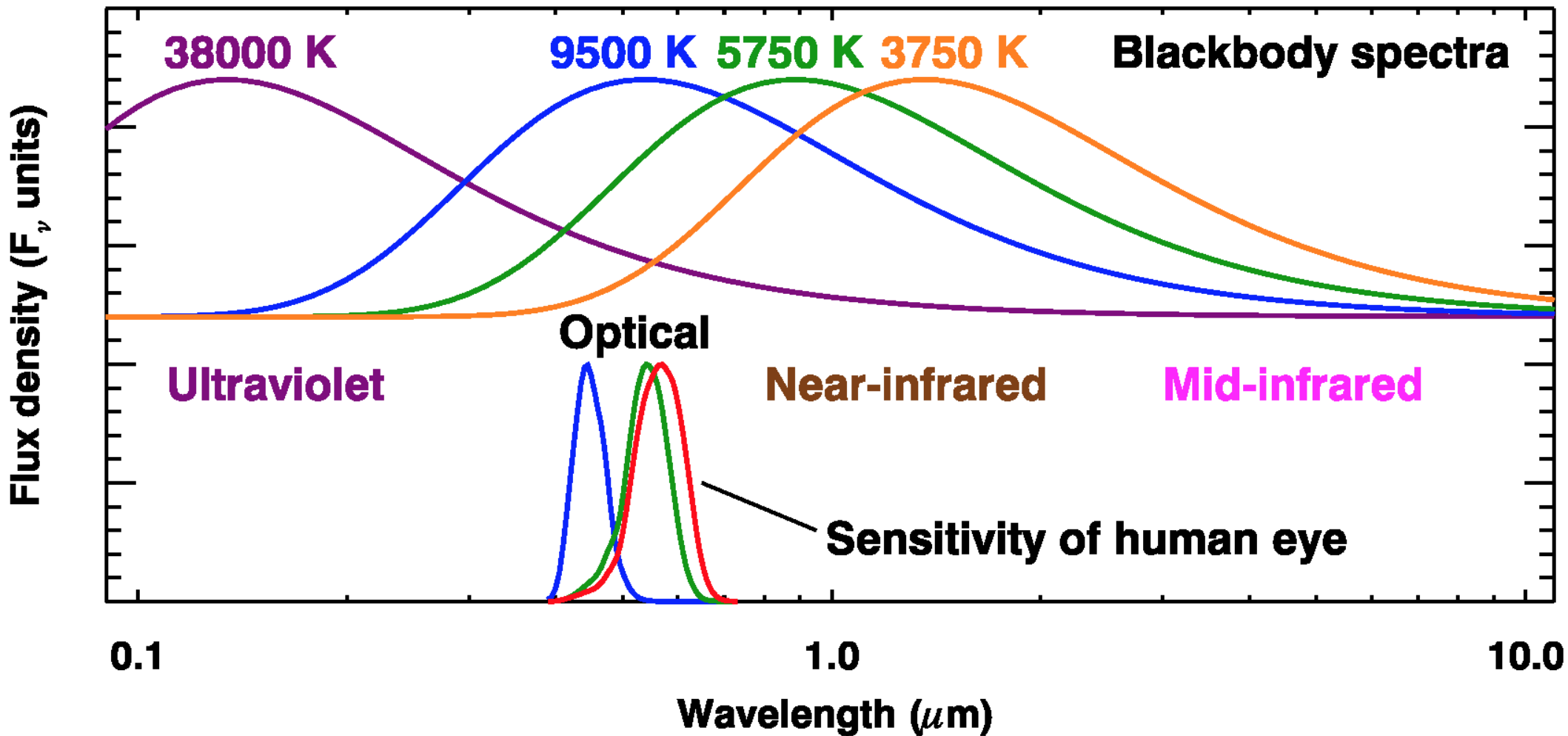
$$y = 5(1 - e^{-y}) \rightarrow \lambda_{\max} T = 0.29 \text{ cm deg}$$

$$\rightarrow \lambda_{\max} = \frac{0.29}{T} \text{ cm deg}$$

Concept: "color" of a star

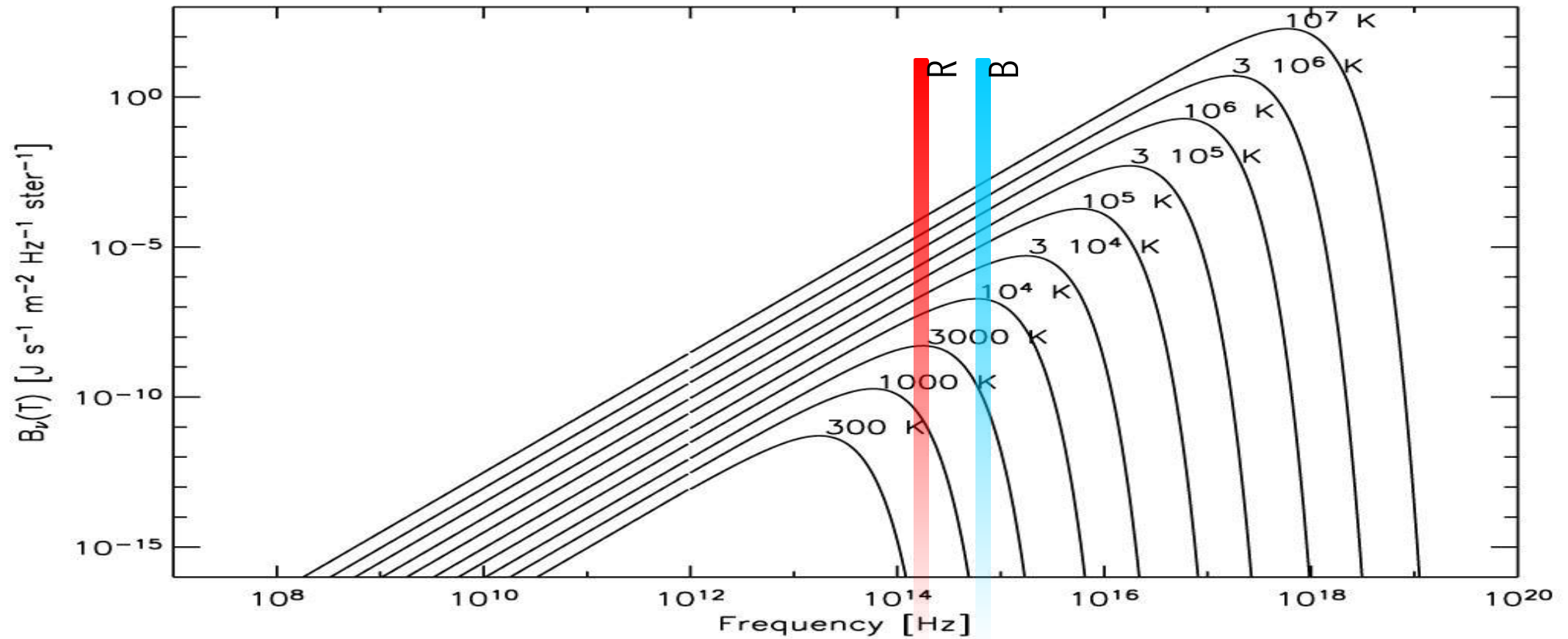


Radiative Transfer & Black Body radiation



Measure the magnitude of a star in two different filters

Compare magnitudes and estimate the temperature



Measure the magnitude of a star in two different filters

Compare magnitudes and estimate the temperature

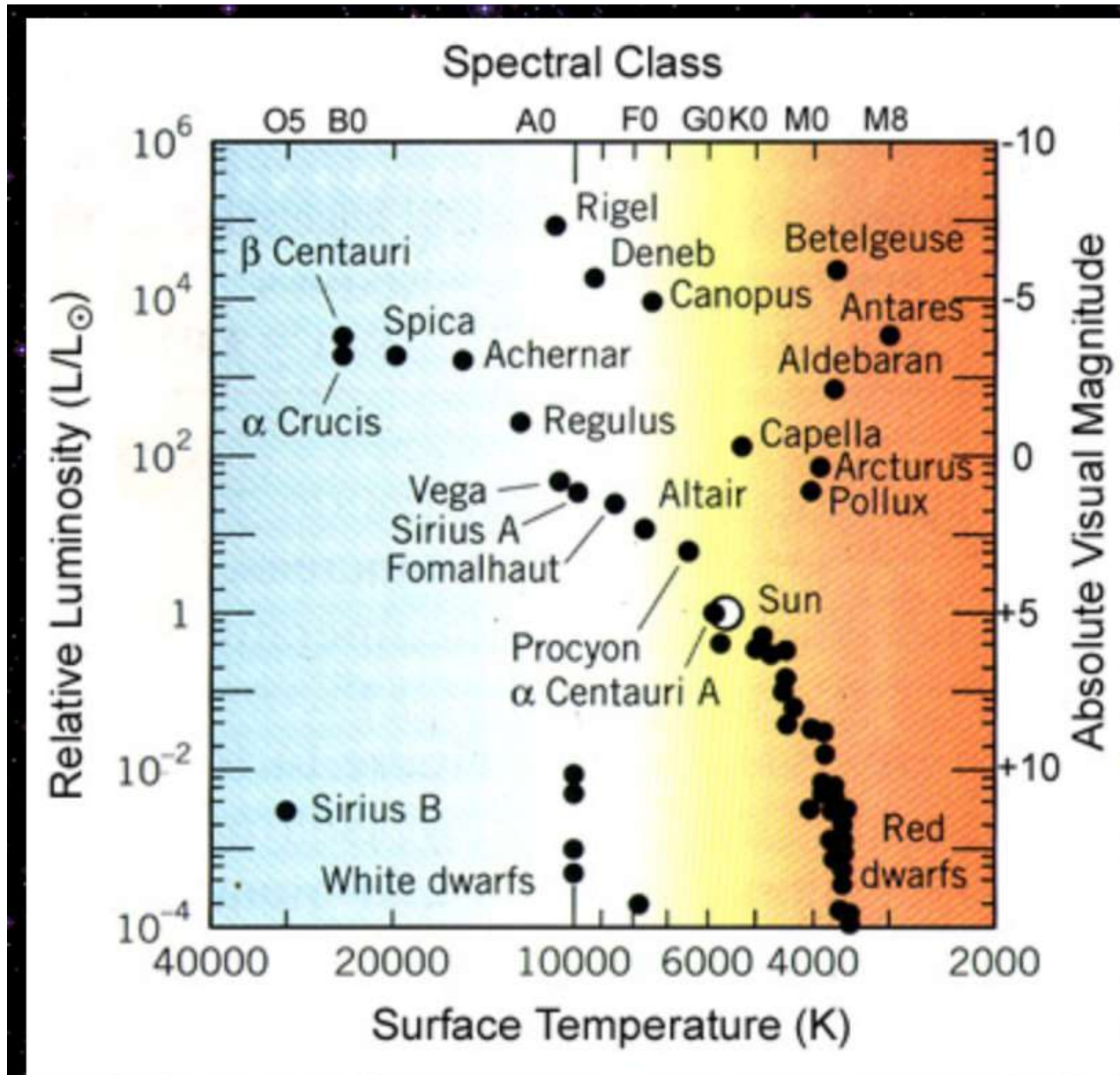


Figure 1. A Limited Hertzsprung-Russell Diagram. (Adapted from Zeilik, *Astronomy: The Evolving Universe*, 8th Ed., Wiley, 1997.)

What kind of info can be derived from the Hertzsprung-Russell Diagram?

What kind of astrophysical bodies can be considered BB emitters?

Stars → UV — NIR
 Dust → IR
 CMB → microwave
 Dense environments → T dependent

Spectra of stars of different temperatures

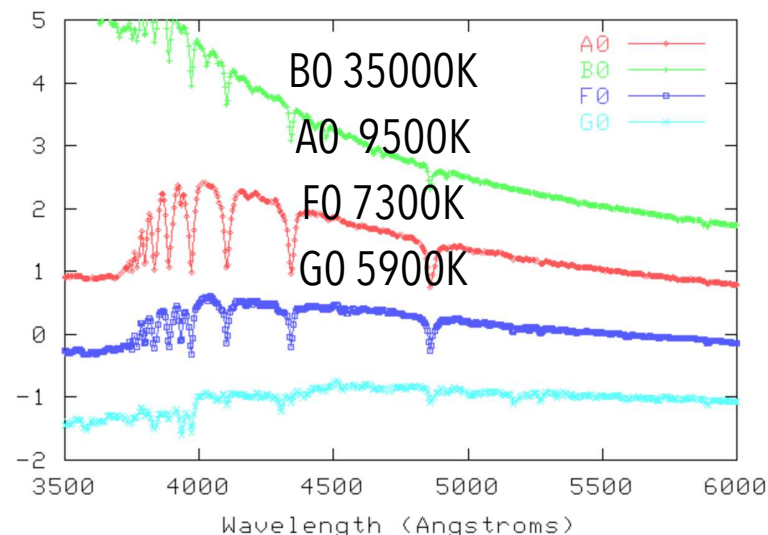
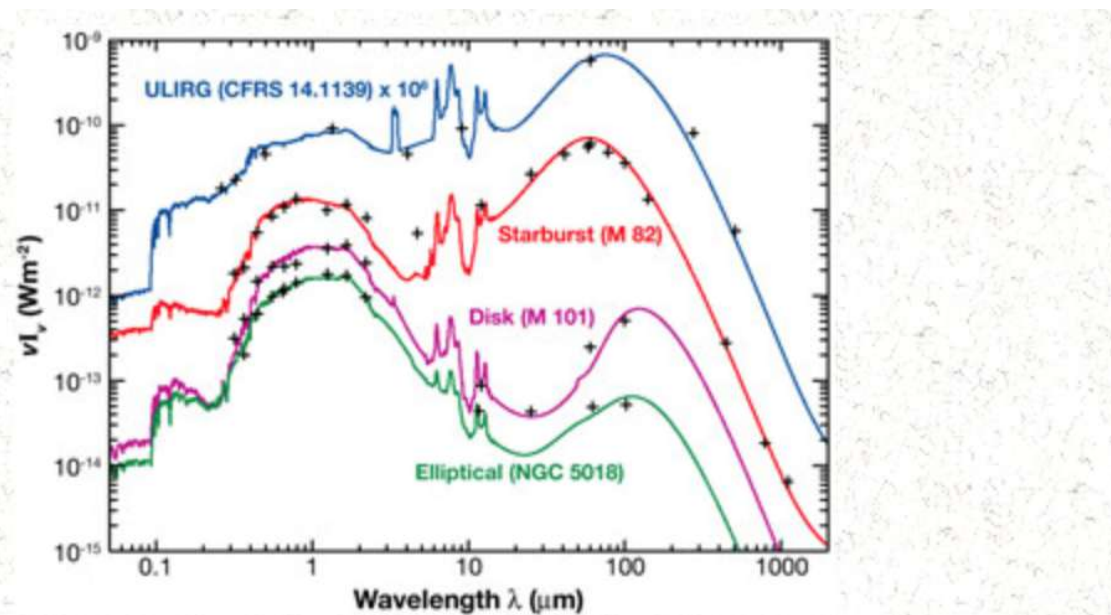
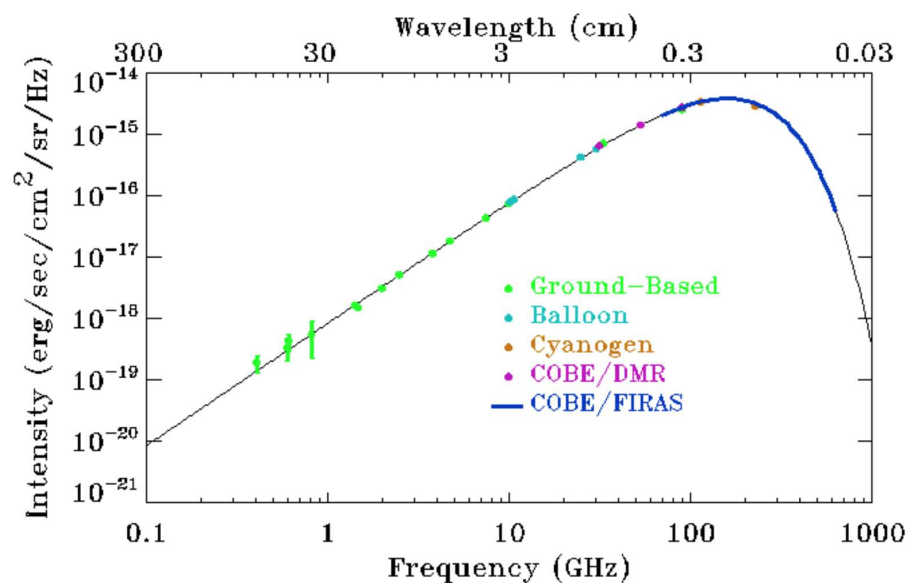
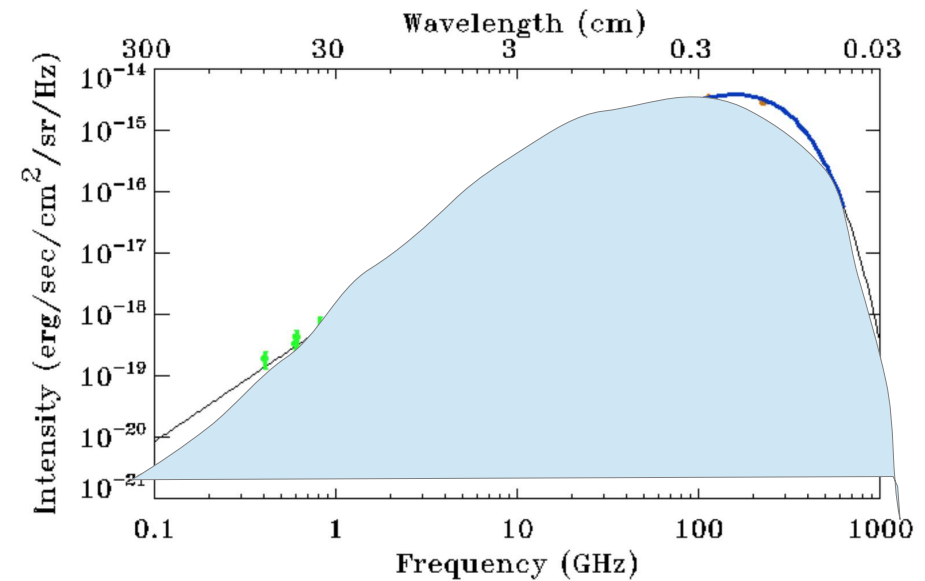


Figure 1. Spectral energy distributions of galaxies from UV to the millimeter. The ULIRG is observed at redshift $z = 0.66$ and is represented here in the rest-frame (from [Galliano 2004](#)).



Bolometric luminosity (Stefan-Boltzman law)

$$L = \sigma T^4$$



Astrophysical application:

Determine the monochromatic luminosity of a star (radius R^* , temperature T^*) in solar Units



Larmor's formula and the acceleration of an electric charge

- Comes from elaboration of Maxwell equations
- Proper maths can be found in textbook (e.g. Rybicki & Lightman, § 2.5, 2.6, 3.1, 3.2, 3.3)
Results are important, algebra is not!
- The interesting part is what happens in astrophysics, where distance is a issue!



In vacuum:

$$\vec{\nabla} \cdot \vec{E} = 0$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$$

let's consider the curl of third eq. and combine it with fourth

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

using the vector identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \vec{\nabla}^2 \vec{E}$

$$\vec{\nabla}^2 \vec{E} - \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2} = 0$$

an identical equation holds for B given that Maxwell equations are invariant for $\vec{E} \rightarrow \vec{B}$, $\vec{B} \rightarrow -\vec{E}$
 solutions are waves whose time averaged Poynting vector is

$$\langle S \rangle = \frac{c}{4\pi} \Re(E_0 B_0^*)$$

FYI,
Computations
never asked



In general: [1] $\vec{\nabla} \cdot \vec{E} = 4\pi\rho_e$ [2] $\vec{\nabla} \cdot \vec{B} = 0$

[3] $\vec{\nabla} \times \vec{E} = -\frac{1}{c} \frac{\partial \vec{B}}{\partial t}$ [4] $\vec{\nabla} \times \vec{B} = \frac{4\pi}{c} \vec{j} + \frac{1}{c} \frac{\partial \vec{E}}{\partial t}$

from [2]: $\vec{B} = \vec{\nabla} \times \vec{A}$ $\rightarrow$ vector potential $\vec{A}(\vec{r}, t)$

then [3] can be written as: $\vec{\nabla} \times \left(\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = 0$

and the argument can be written as a gradient of a scalar field $\vec{\nabla} \phi$

$$\vec{E} + \frac{1}{c} \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \phi$$

$$\vec{E} = -\frac{1}{c} \frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi$$

equation [1] becomes $\nabla^2 \phi + \frac{1}{c} \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) = -4\pi\rho_e$

equation [4] is $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \frac{1}{c} \frac{\partial}{\partial t} \left(-\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = -\frac{4\pi}{c} \vec{j}$



using the vector identity $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \vec{\nabla}^2 \vec{A}$

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) - \frac{1}{c} \frac{\partial}{\partial t} \left(-\vec{\nabla} \phi - \frac{1}{c} \frac{\partial \vec{A}}{\partial t} \right) = -\frac{4\pi}{c} \vec{j} \quad \text{becomes}$$
$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} \right) = -\frac{4\pi}{c} \vec{j}$$

The potentials ϕ and $\vec{A}$ are not uniquely determined given that

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla} \psi \quad \text{implies} \quad \vec{B} \rightarrow \vec{B}$$

$$\phi \rightarrow \phi - \frac{1}{c} \frac{\partial \psi}{\partial t} \quad \text{implies} \quad \vec{E} \rightarrow \vec{E}$$

modifications of ϕ and $\vec{A}$ are known as **Gauge transformations**; ψ is a scalar function

The **Lorentz gauge** satisfies the Lorentz condition:

$$\vec{\nabla} \cdot \vec{A} + \frac{1}{c} \frac{\partial \phi}{\partial t} = 0$$

which greatly simplifies the above equations

The two potentials can be simplified (Lorentz gauge):

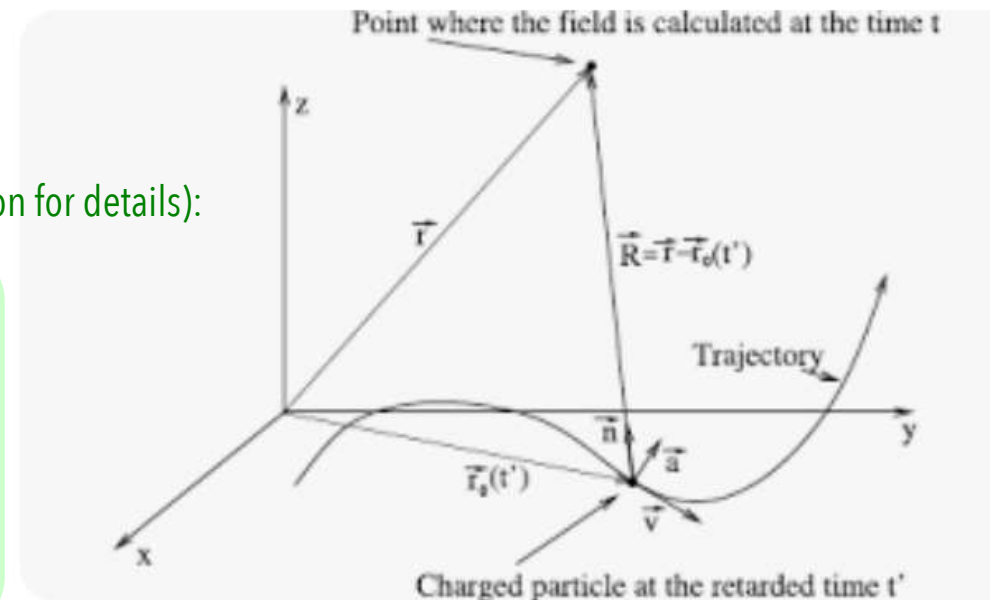
$$\vec{\nabla}^2 \phi - \frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} = -\frac{4\pi}{c} \rho_e$$

$$\vec{\nabla}^2 \vec{A} - \frac{1}{c^2} \frac{\partial^2 \vec{A}}{\partial t^2} = -\frac{4\pi}{c} \vec{j}$$

solutions have the following expression (see Jackson for details):

$$\phi(\vec{r}, t) = \int \rho_e \frac{d^3 \vec{r}'}{|\vec{r} - \vec{r}'|}$$

$$\vec{A}(\vec{r}, t) = \frac{1}{c} \int \vec{j} \frac{d^3 \vec{r}'}{|\vec{r} - \vec{r}'|}$$



The integrals are computed over the source (volume) defining ρ_e and $\vec{j}$

They are known as **retarded potentials** given they consider a **time delay** required for light to travel from the origin of the potential (charge and current) to the observer

A charge q is moving along a given trajectory $\vec{r} = \vec{r}_0(t)$ with velocity $\vec{u} = \dot{\vec{r}}_0(t)$
 at a given position $\vec{r}$ we get:

$$\rho_e(\vec{r}, t) = q \delta[\vec{r} - \vec{r}_0(t)]$$

$$\vec{j}(\vec{r}, t) = q \vec{u}(t) \delta[\vec{r} - \vec{r}_0(t)]$$

and

$$q = \int \rho_e(\vec{r}, t) d^3 \vec{r}$$

$$q \vec{u} = \int \vec{j}(\vec{r}, t) d^3 \vec{r}$$

the scalar potential is

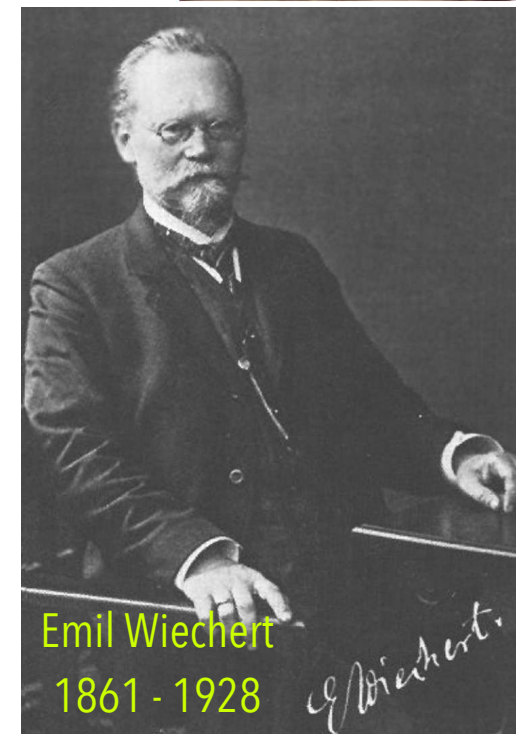
$$\phi(\vec{r}, t) = \int d^3 \vec{r}' \int dt' \frac{\rho_e(\vec{r}', t')}{|\vec{r} - \vec{r}'|} \delta\left(t' - t + \frac{|\vec{r} - \vec{r}'|}{c}\right)$$

and inserting the above definition of charge density and integrating over distance

$$\phi(\vec{r}, t) = q \int \delta\left(t' - t + \frac{|\vec{r} - \vec{r}_0(t')|}{c}\right) \frac{dt'}{|\vec{r} - \vec{r}_0(t')|}$$

if we introduce $\vec{R}(t') = |\vec{r} - \vec{r}_0(t')| \rightarrow R(t') = |\vec{R}(t')|$

and consider the vector potential as well we get



the potentials are

$$\phi(\vec{r}, t) = q \int R^{-1}(t') \delta\left(t' - t + \frac{R(t')}{c}\right) dt'$$

$$\vec{A}(\vec{r}, t) = \frac{q}{c} \int \vec{u}(t') R^{-1}(t') \delta\left(t' - t + \frac{R(t')}{c}\right) dt'$$

In particular if we consider $t' = t_{\text{ret}}$ so that $R(t_{\text{ret}}) = c(t - t_{\text{ret}})$

and change variable to $t'' = t' - t + R(t')/c$

whose increment becomes $dt'' = dt' + \frac{1}{c} \dot{R}(t') dt'$

If we consider $R^2(t') = \vec{R}(t') \cdot \vec{R}(t')$ and take the time derivative

$$2R(t')\dot{R}(t') = \dot{\vec{R}}(t') \cdot \vec{R}(t') + \vec{R}(t') \cdot \dot{\vec{R}}(t') = -2\vec{R}(t') \cdot \dot{\vec{R}}(t')$$

namely $\dot{\vec{R}}(t') = -\vec{u}(t')$ finally we introduce $\hat{n} = \frac{\vec{R}}{R}$

the increment can then be written as $dt'' = [1 - \frac{1}{c} \vec{n}(t') \cdot \vec{u}(t')] dt'$

$$[1 - \frac{1}{c} \vec{n}(t') \cdot \vec{u}(t')]^{-1} dt'' = dt'$$

and the potentials become $\phi(\vec{r}, t) = q \int R^{-1}(t') [1 - \frac{1}{c} \vec{n}(t') \cdot \vec{u}(t')]^{-1} \delta(t'') dt''$

$$\vec{A}(\vec{r}, t) = \frac{q}{c} \int \vec{u}(t') R^{-1}(t') [1 - \frac{1}{c} \vec{n}(t') \cdot \vec{u}(t')]^{-1} \delta(t'') dt''$$

the delta function can be integrated if $t'' = 0$ (or equivalently $t' = t_{\text{ret}}$) yielding the

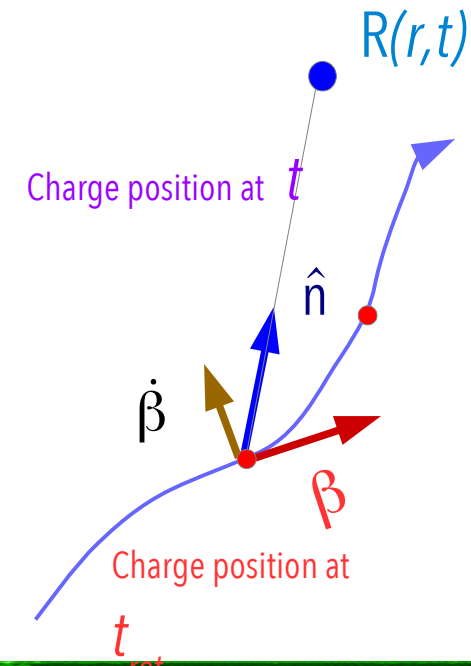
Lienard - Wiechart potentials

$$\phi(\vec{r}, t) = \frac{q}{R(t_{\text{ret}}) \kappa(t_{\text{ret}})}$$

$$\vec{A}(\vec{r}, t) = \frac{q \vec{u}}{c R(t_{\text{ret}}) \kappa(t_{\text{ret}})}$$

where $\kappa(t') = 1 - \frac{1}{c} \vec{n}(t') \cdot \vec{u}(t')$ $[t' = t_{\text{ret}}]$

→ difference wrt static electromagnetic theory, relevant when $u \sim c$
→ concentrates the potentials about the particle velocity (**beaming**)





Radiation from moving charges

Differentiation of the Liénard-Wiechert potentials (easy but lengthy) produces the radiation fields at a position $\vec{r}$ and a time t (computed at the "retarded time" t_{ret} and corresponding position $\vec{r}_{\text{ret}}$)

Let $\vec{\beta} \stackrel{\text{def}}{=} \frac{\vec{u}}{c}$ and then $\kappa \stackrel{\text{def}}{=} 1 - \vec{n} \cdot \vec{\beta}$

$1/R$, acceleration field
 $\perp$ to $\vec{n} \rightarrow$ radiation field

$$\vec{E}(\vec{r}, t) = q \underbrace{\left(\frac{(\vec{n} - \vec{\beta})(1 - \beta^2)}{\kappa^3 R^2} \right)}_{1/R^2, \text{ velocity field}} + \frac{q}{c} \left(\frac{\vec{n}}{\kappa^3 R} \times [(\vec{n} - \vec{\beta}) \times \vec{\beta}] \right)$$

$1/R^2$, velocity field
 generalization of Coulomb's law

$$\vec{B}(\vec{r}, t) = \vec{n} \times \vec{E}(\vec{r}, t)$$

1. Coulomb's law holds for $\beta \ll 1$ & no acceleration; $\vec{E}$ field points to current position of the charge
2. In case of acceleration the radiation field [at large distances from the charge] is then

$$\vec{E}_{\text{rad}}(\vec{r}, t) = \frac{q}{c} \left(\frac{\vec{n}}{\kappa^3 R} \times [(\vec{n} - \vec{\beta}) \times \vec{\beta}] \right)$$

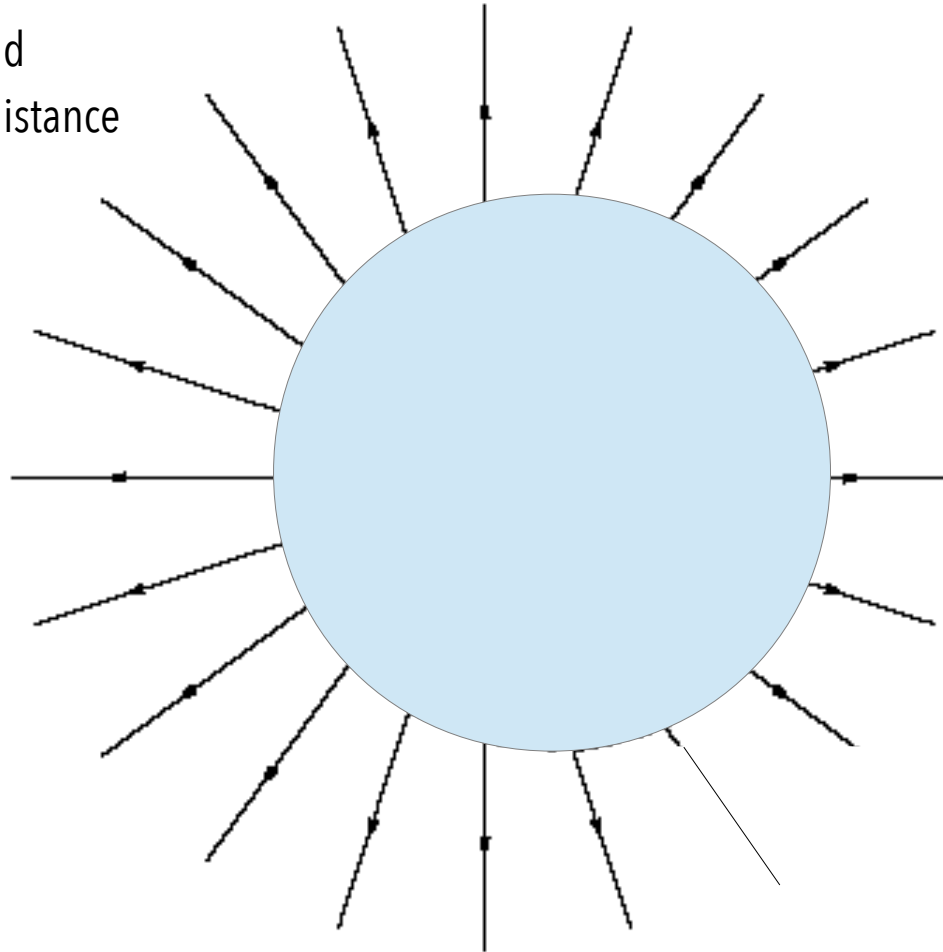
$$\vec{B}_{\text{rad}}(\vec{r}, t) = \vec{n} \times \vec{E}_{\text{rad}}(\vec{r}, t)$$



Let's make a test

The black lines represent the E field
of a charge q , measured at large distance

Question:
where is the charge q (now)?



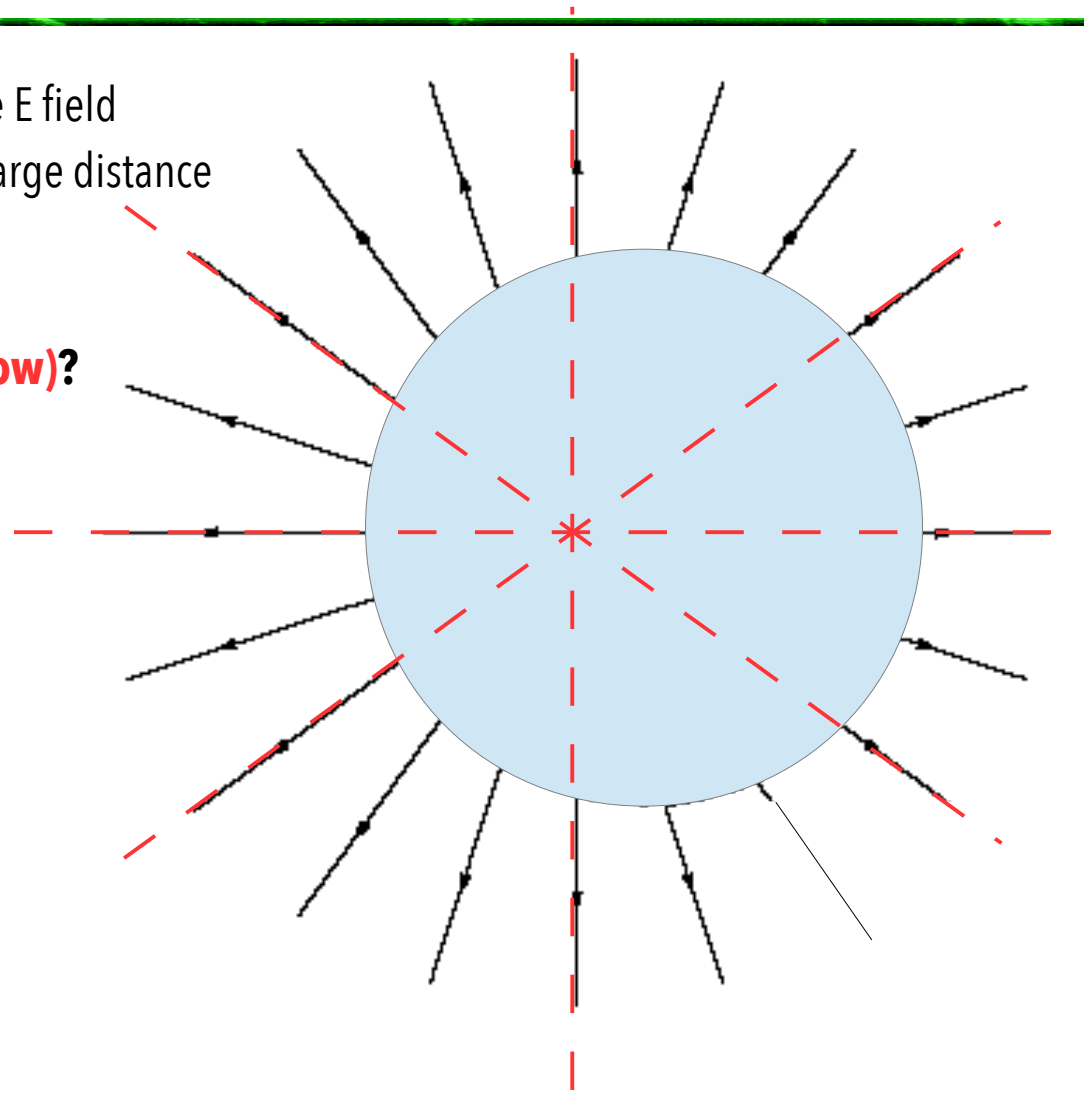


Let's make a test

The black lines represent the E field
of a charge q , measured at large distance

Question:
where is the charge q (now)?

Easy solution!





Let's make a test

The black lines represent the E field of a charge q , measured at large distance

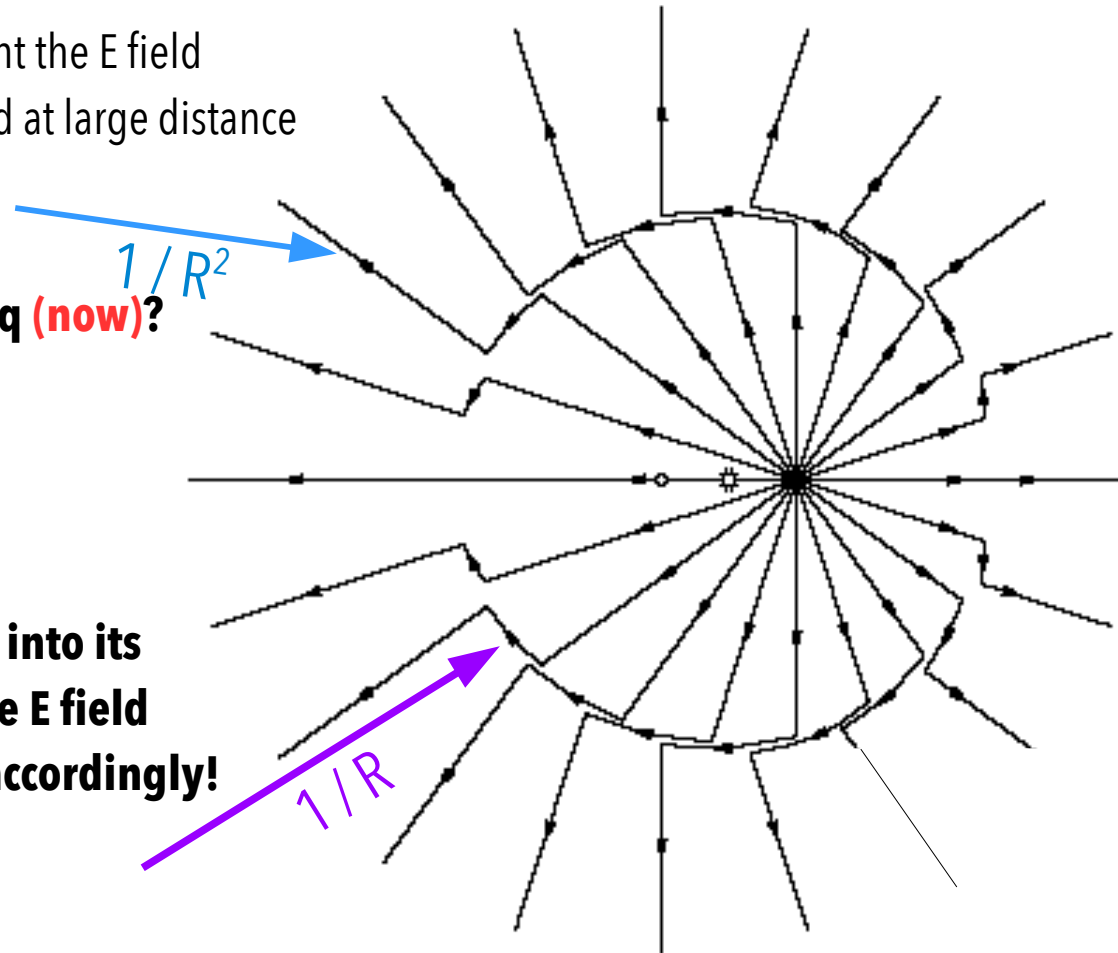
Question:

where is the charge q (now)?

Easy solution!

But can be wrong!

The charge q moved into its new position and the E field has been modified accordingly!



See <https://www.youtube.com/watch?v=a5X5EsX9YmE>

$\mathbf{E}_{rad} - \mathbf{B}_{rad} - \mathbf{n}$ right-hand system of perpendicular vectors, with $|\mathbf{E}_{rad}| = |\mathbf{B}_{rad}|$
and are consistent with radiation solutions of Maxwell equations



Radiation from moving charges (2)

Comparison between the two components of the $\mathbf{E}$ [B] field [if $\beta \ll 1$]

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} \sim \frac{R \dot{u}}{c^2}$$

and for a given characteristic oscillation ν then $\dot{u} = u \nu$

$$\frac{E_{\text{rad}}}{E_{\text{vel}}} \sim \frac{R u \nu}{c^2} = \frac{u R}{c \lambda}$$

Two zones can be highlighted :

1. "**near zone**", where $R \leq \lambda$, and the $E_{\text{vel}} > E_{\text{rad}}$ by a factor c / u ;

Coulomb effects dominate

2. "**far zone**" or "**wave zone**", at large $R \geq (\lambda c / u)$, where E_{rad} dominates and whose dominance progressively increases with R ;

radiation is the only thing that matters → relevant to astronomy!

If a charge is accelerated ($\vec{u} \neq 0$) and $\beta \ll 1$ fields are:

$$\vec{E}_{\text{rad}}(\vec{r}, t) = \frac{q}{c} \left(\frac{\vec{n}}{R^3} \times [(\vec{n} - \beta) \times \dot{\beta}] \right) \simeq \frac{q}{Rc^2} [\vec{n} \times (\vec{n} \times \vec{u})]$$

$$\vec{B}_{\text{rad}}(\vec{r}, t) = \vec{n} \times \vec{E}_{\text{rad}}(\vec{r}, t)$$

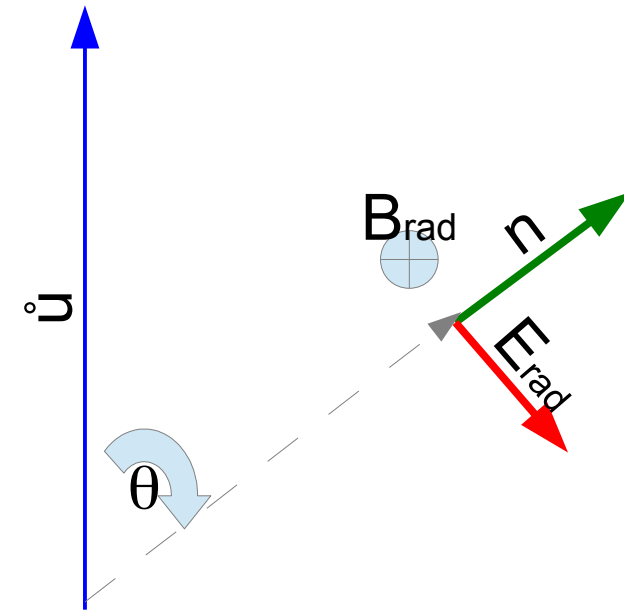
$\vec{E}_{\text{rad}}$ is perpendicular to the acceleration of the particle

At a given point the maximum field amplitudes can be computed

$$|\vec{E}_{\text{rad}}| = |\vec{B}_{\text{rad}}| = \frac{q\dot{u}}{Rc^2} \sin \theta$$

The Poynting vector providing the emitted power is then

$$\begin{aligned} \langle S \rangle &= \frac{c}{4\pi} |\vec{E}_{\text{rad}}|^2 = \frac{c}{4\pi} \left(\frac{q\dot{u}}{Rc^2} \sin \theta \right)^2 = \\ &= \frac{q^2 \dot{u}^2}{4\pi c^3 R^2} \sin^2 \theta = \frac{dW}{dt d\Omega} \end{aligned}$$





Integrating over the solid angle we get the total emitted power

$$P = \frac{dW}{dt} = \frac{q^2 \dot{u}^2}{4\pi c^3} \int \sin^2 \theta d\Omega = \frac{2q^2 \dot{u}^2}{3c^3}$$

Larmor's Formula

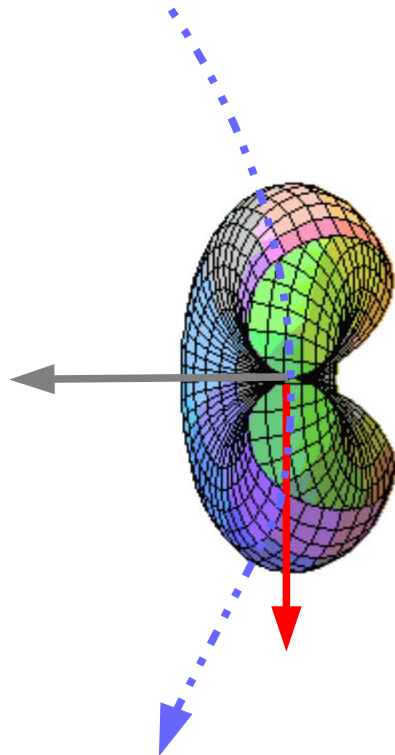
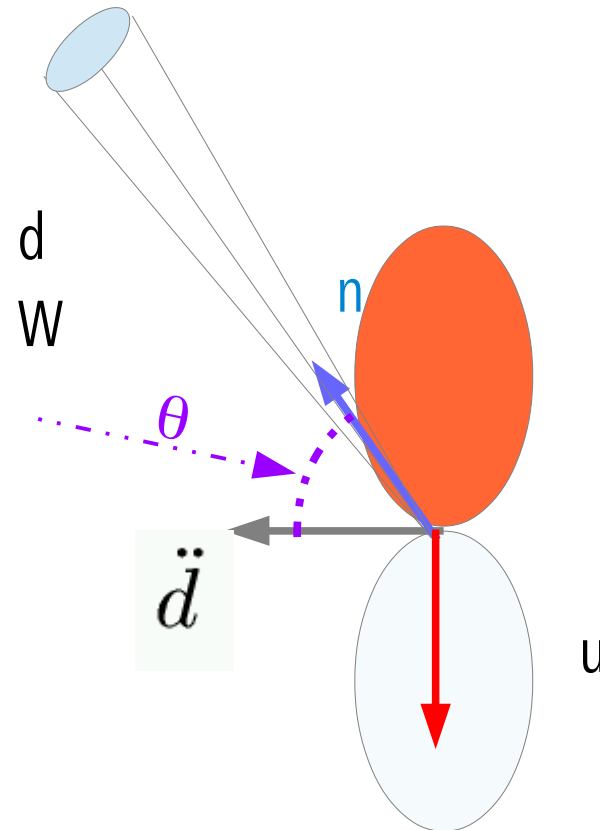
Dipole approximation of Larmor's formula

$$P = \frac{dW}{dt} = \frac{2q^2 a^2}{3c^3} = \frac{2q^2 \dot{p}^2}{3m^2 c^3} = \frac{2}{3} \frac{\ddot{\theta}^2}{c^3}$$

Let's consider the angular distribution i.e. how an observer gets the emitted radiation

$$S = \frac{c}{4\pi} E_{\text{rad}}^2 = \frac{c}{4\pi} \frac{q^2 \dot{u}^2}{R^2 c^2} \sin^2 \theta = \frac{dW}{dt R^2 d\Omega}$$

Dipole radiation pattern, $l=1, m=0$





Relativistic charge: ($\gamma \gg 1$)

We must express the Larmor's formula in an invariant form: scalars are not affected; let's write

$$d\tau = \frac{dt}{\gamma}$$

$$p_i = [\vec{p}, \frac{i}{c}W]$$

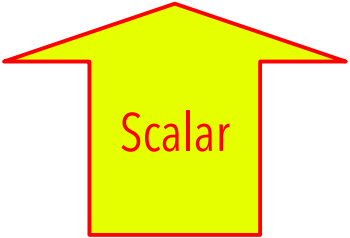
$$W^2 = p^2 c^2 + m^2 c^4 \quad (p/W = v/c^2)$$

$$P = - \frac{dW}{dt} = \frac{2q^2}{3m^2 c^3} \left[\frac{dp_i}{d\tau} \frac{dp_i}{d\tau} \right]$$

$$\left[\frac{dp_i}{d\tau} \frac{dp_i}{d\tau} \right] = \left(\frac{d\vec{p}}{d\tau} \right)^2 - \frac{1}{c^2} \left(\frac{dW}{d\tau} \right)^2 = \left(\frac{d\vec{p}}{d\tau} \right)^2 - \beta^2 \left(\frac{dp}{d\tau} \right)^2$$



Vector



Scalar



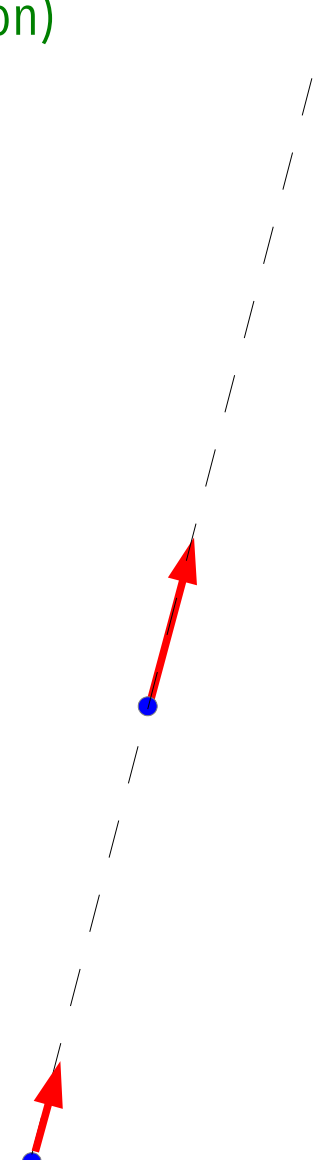
Linear acceleration ($\vec{p}$ does not change direction, no centripetal acceleration)

$$\left| \frac{d\vec{p}}{d\tau} \right| \simeq \frac{dp}{d\tau}$$

$$\left(\frac{d\vec{p}}{d\tau} \right)^2 - \beta^2 \left(\frac{dp}{d\tau} \right)^2 \simeq \left(\frac{dp}{d\tau} \right)^2 - \beta^2 \left(\frac{dp}{d\tau} \right)^2 \simeq \frac{1}{\gamma^2} \left(\frac{dp}{d\tau} \right)^2$$

$$P = -\frac{dW}{dt} = \frac{2q^2}{3m^2c^3} \frac{1}{\gamma^2} \left(\frac{dp}{d\tau} \right)^2 \simeq \frac{2q^2}{3m^2c^3} \left(\frac{dp}{dt} \right)^2$$

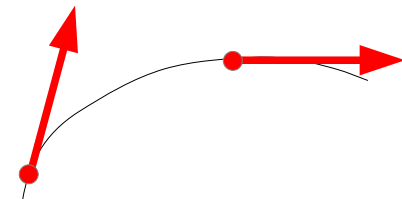
The same as the non-relativistic case!



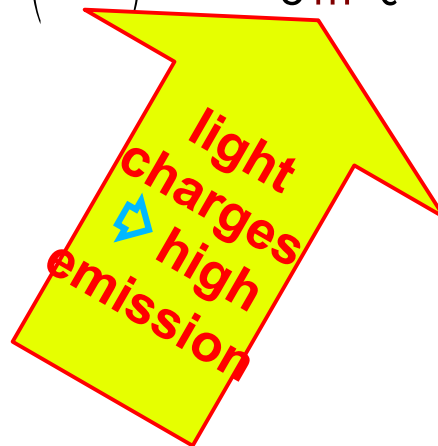
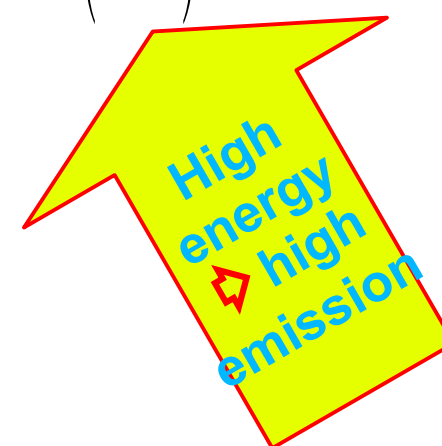
Centripetal acceleration (the change in direction largely exceeds the change in velocity; the energy does (may) not change substantially during the interaction)

$$\left| \frac{d\vec{p}}{d\tau} \right| \gg \beta \left(\frac{dp}{d\tau} \right) = \frac{1}{c} \left(\frac{dW}{d\tau} \right)$$

$$\left[\frac{dp_i}{d\tau} \frac{dp_i}{d\tau} \right] = \left(\frac{d\vec{p}}{d\tau} \right)^2 - \frac{1}{c^2} \left(\frac{dW}{d\tau} \right)^2 \approx \left(\frac{d\vec{p}}{d\tau} \right)^2 - \cancel{\beta^2 \left(\frac{dp}{d\tau} \right)^2}$$



$$P = -\frac{dW}{dt} = \frac{2q^2}{3m^2c^3} \left(\frac{d\vec{p}}{d\tau} \right)^2 \approx \frac{2q^2}{3m^2c^3} \gamma^2 \left(\frac{d\vec{p}}{dt} \right)^2$$





$$P = -\frac{dW}{dt} = \frac{2q^2}{3m^2c^3} \left(\frac{d\vec{p}}{d\tau} \right)^2 \simeq \frac{2q^2}{3m^2c^3} \gamma^2 \left(\frac{d\vec{p}}{dt} \right)^2$$

Important notes

— 1. $P \simeq \gamma^2 \rightarrow$ **The most energetic charges are the most effective emitters**

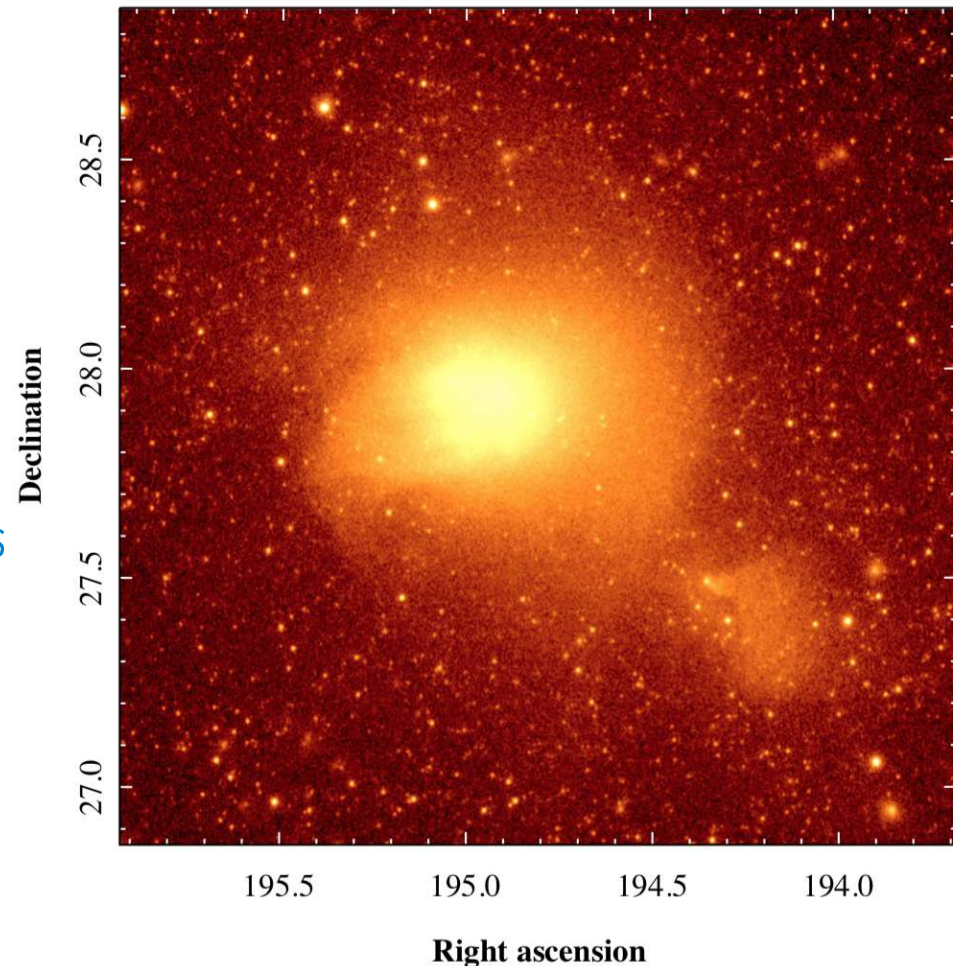
— 2. $P \simeq m^{-2} \rightarrow$ **The lightest charges are the most effective emitters**

Electrons (positrons) radiate $\sim 4 \cdot 10^6$ more power than protons with the same energy (γ)

- Interaction (near collision) between electron and ion
- The Coulomb force is at work
- Let's consider a reference frame at rest with the ion
- The electron releases a photon (energy) at expenses of its own kinetic energy

Outline

- Single event
- An unrealistic example
- A thermal plasma (OK for astrophysics)



E-Rosita X-ray image of the Coma cluster of galaxies
Churazov et al. 2021

G. Ghisellini: "Radiative Processes in High Energy Astrophysics" § 2

M. Longair: "High Energy Astrophysics" § 6.3 – 6.6

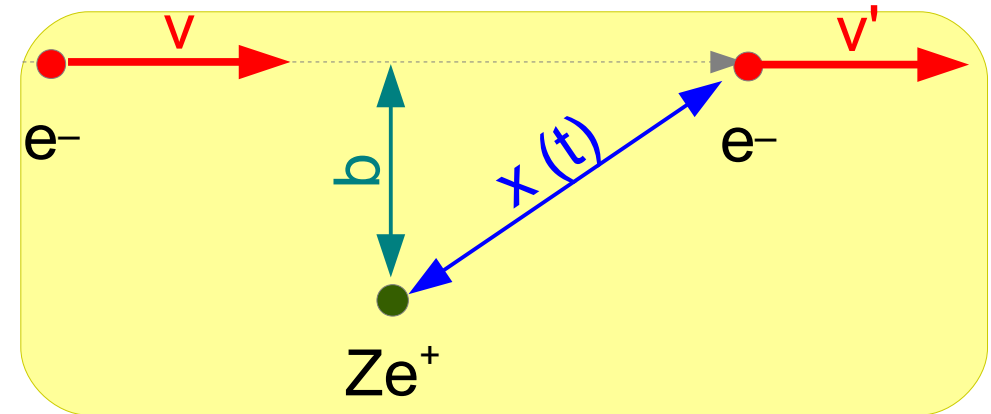
G. B. Rybicki & A.P. Lightman: "Radiative Processes in Astrophysics" § 5

Radiated power (single particle) is given by the Larmor's formula:

$$P = -\frac{dE}{dt} = \frac{2e^2}{3c^3} a^2$$

acceleration is from Coulomb's law

$$F = ma \simeq -\frac{Ze^2}{x^2} \rightarrow a = \frac{Ze^2}{mx^2}$$



x = actual distance between electron (e^-) and ion (Ze^+)

more appropriately $x \sim x(t)$ then

$$P(t) = \frac{2e^2}{3c^3} a^2(t) = \frac{2Z^2e^6}{3c^3} \frac{1}{m^2} \frac{1}{[x(t)]^4}$$

Remarks:

1. $a \sim m^{-1} \rightarrow P \sim m^{-2}$
2. $a \sim x^{-2} \rightarrow P \sim x^{-4}$

Radiated power is relevant

1. for electrons (positrons) only i.e. light particles
2. when the two charges are at the closest distance: ($x \sim b$)

The interaction lasts very shortly: $\Delta t \sim \frac{2b}{v}$ where $v =$ velocity

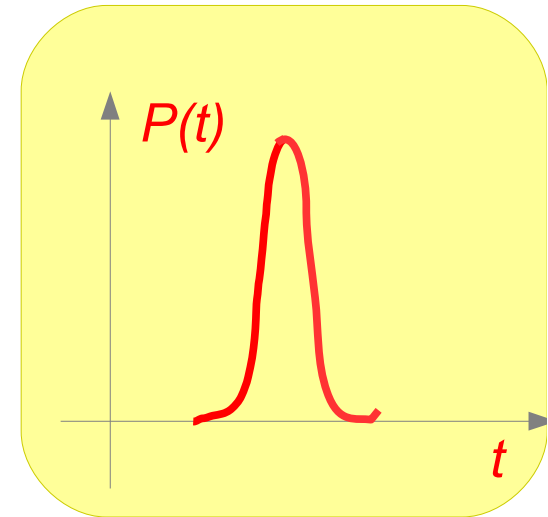
The emitted energy in a single "collision" (consider $x \sim b$)

$$P \Delta t = \frac{2}{3} \frac{e^2}{c^3} \left(\frac{Ze^2}{mx^2} \right)^2 \frac{2b}{v} = \frac{4}{3} \frac{Z^2 e^6}{c^3 m^2} \frac{1}{b^3 v}$$

$x \simeq b$

Radiation comes in pulses Δt long

Asymmetric profile (decrease of the e- velocity)

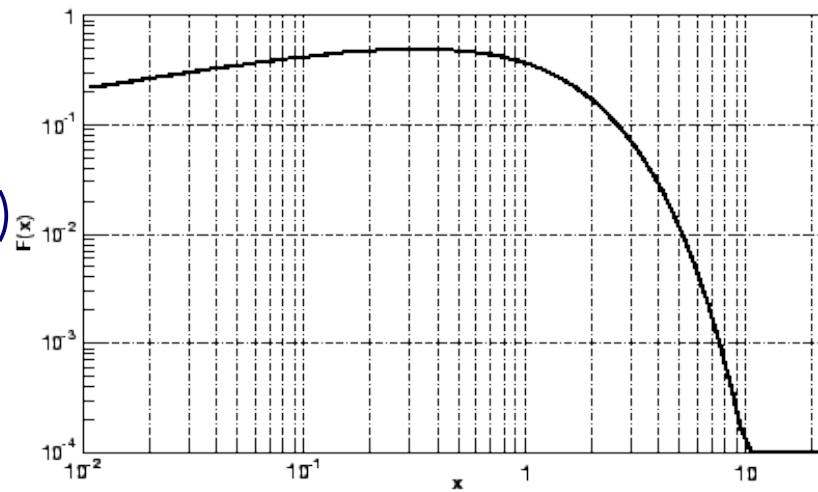
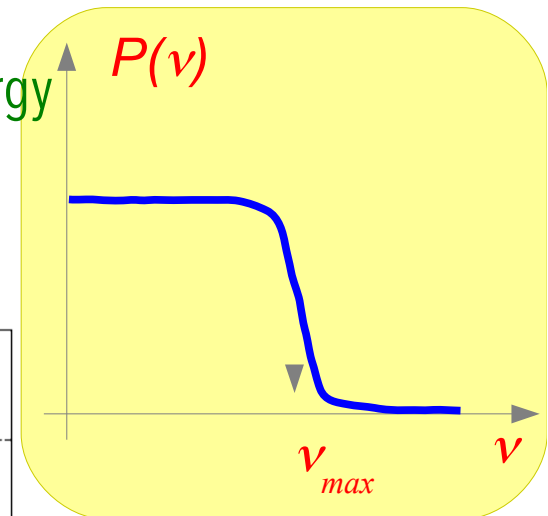


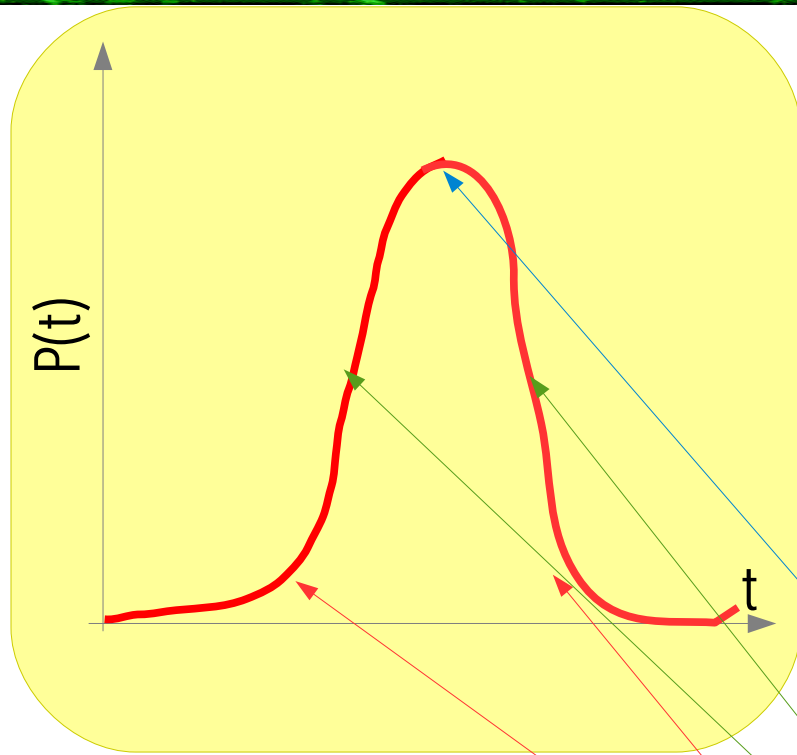
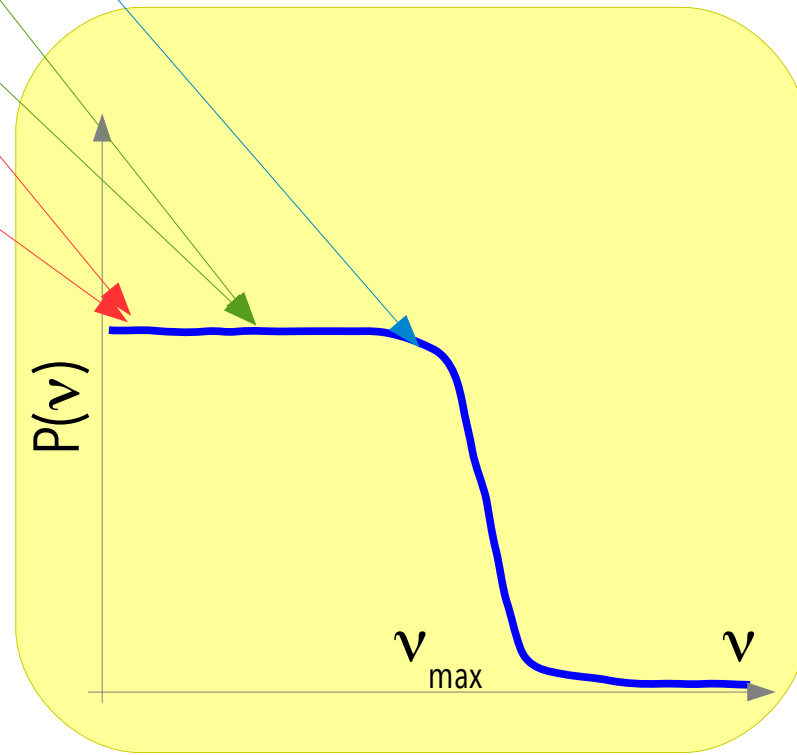
Fourier analysis of the pulse $\rightarrow$ Spectral distribution of the radiated energy

Flat up to a frequency $\nu_{\max} \simeq \frac{1}{2\Delta t} \approx \frac{v}{4b}$

Related to the kinetic energy of the electrons

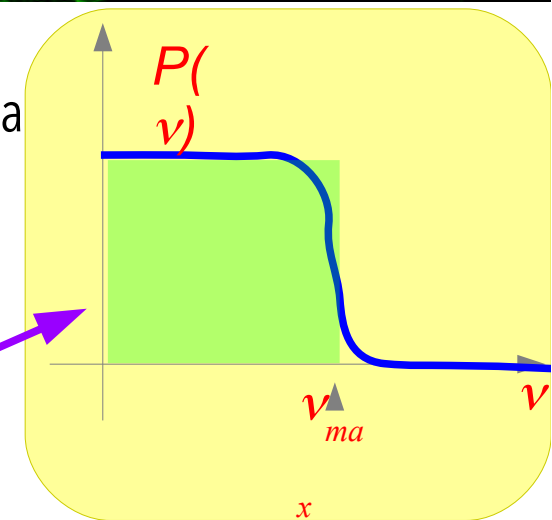
(exponential decay beyond $\nu_{\max}$)





The radiated energy per unit frequency (assuming flat spectrum) from a single event can be written as

$$\frac{P \Delta t}{\Delta \nu} \approx \frac{P \Delta t}{\nu_{\max}} = 2 P (\Delta t)^2 \simeq \frac{16}{3} \frac{Z^2 e^6}{c^3 m_e^2} \frac{1}{b^2 \nu^2} \approx \frac{1}{b^2 \nu^2}$$



Exercise:

Determine the range of distances (order of mag) in which the released energy is significant

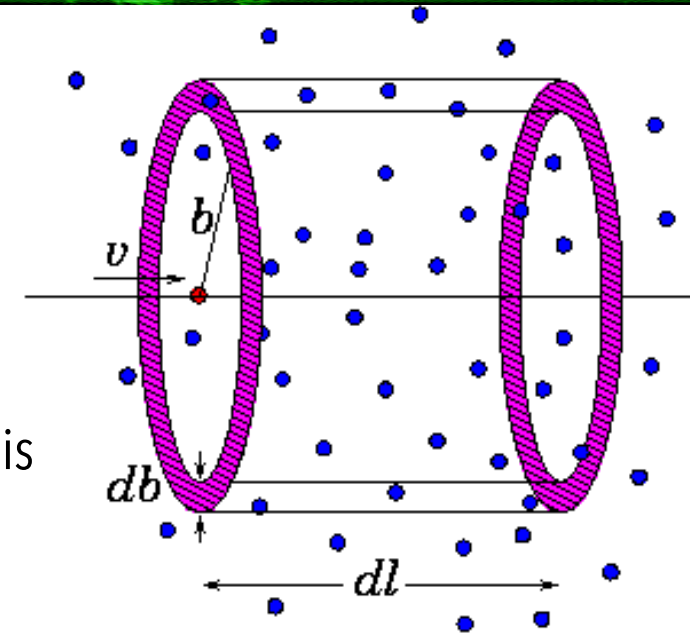
$$e = 4.8 \cdot 10^{-10} \text{ statC} \quad m_e = 9 \cdot 10^{-28} \text{ g} \quad v = 100 \text{ km/s}$$

$$\text{Remember.... Bohr's radius} \quad a_0 \sim 5 \cdot 10^{-13} \text{ cm}$$

$$\text{Typical ISM densities} \quad n_{\text{ISM}} \sim 1 \text{ cm}^{-3}$$

Unphysical case: a cloud with ion number density n_Z and free electron number density n_e

In a unit time, **each electron** experiences a number of collisions with an impact parameter between b and $b + db$ is

$$2\pi n_Z v b db$$


The total number of collisions is $2\pi n_e n_Z v b db$

Total emissivity from a cloud emitting via bremsstrahlung (single velocity!)

$$J_{br}(v, \nu) = \frac{dE}{d\nu dV dt} = 2\pi n_e n_Z v \int_{b_{min}}^{b_{max}} \frac{16 Z^2 e^6}{3 c^3 m_e^2} \frac{1}{b^2 v^2} b db =$$

$$= \frac{32 \pi e^6}{3 c^3 m_e^2 v} n_e n_Z Z^2 \int_{b_{min}}^{b_{max}} \frac{db}{b} = \frac{32 \pi e^6}{3 c^3 m_e^2} \frac{1}{v} n_e n_Z Z^2 \ln\left(\frac{b_{max}}{b_{min}}\right)$$

Evaluation of $b_{\max}$:

at a given ν , we must consider only those interactions where electrons have impact parameters corresponding to $\nu_{\max} > \nu$ and then

$$b_{\max} \leq \frac{v}{4\nu} \quad \text{is a function of frequency}$$

Evaluation of $b_{\min}$:

a) classical physics requires that $\Delta v \leq v$

$$\Delta v \simeq a \Delta t = \frac{Ze^2}{m_e b^2} \frac{2b}{v} \leq v \quad \rightarrow \quad b_{\min-c} \geq \frac{2Ze^2}{m_e v^2}$$

b) quantum physics and the Heisenberg principle $\Delta p \Delta x \geq h/2\pi$

$$\Delta p = m_e \Delta v \approx m_e v \geq \frac{h}{2\pi \Delta x} \approx \frac{h}{2\pi b_{\min}} \quad \rightarrow \quad b_{\min-q} \geq \frac{h}{2\pi m_e v}$$



we must consider which $b_{\min}$ has to be used

A set of particles at thermal equilibrium follow Maxwell-Boltzmann distribution:

$$f(v)dv = 4\pi \left(\frac{m_e}{2\pi kT} \right)^{3/2} e^{-m_e v^2 / 2kT} v^2 dv$$

Then $n_e(v) = n_e f(v)dv$ is the number density of electrons whose velocity is between v and $v+dv$ and $n_e(v)$ replaces n_e in $J_{br}(v)$

When the plasma is at equilibrium, the process is termed **thermal bremsstrahlung**

Total emissivity:

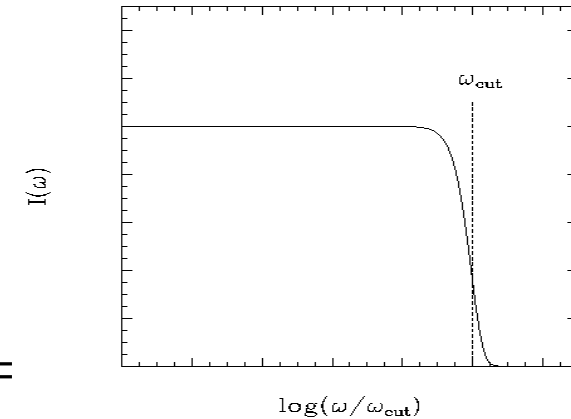
must integrate over velocity (energy) distribution ($0 \leq v < \infty$?)

$$J_{br}(v, T) = \frac{dE}{dt dV dv} = \int_{v_{min}}^{\infty} J_{br}(v, v) f(v) dv =$$

$$= 6.8 \cdot 10^{-38} T^{-1/2} e^{-h\nu/kT} n_e n_Z Z^2 g_{ff}(v, T) \text{ erg s}^{-1} \text{ cm}^{-3} \text{ Hz}^{-1}$$

Plasma density (n_e^2)

T ~ particle energy
high energy cut-off (T)



$$b_{\max} \simeq \frac{v}{4\nu} \quad \text{frequency dependent!....velocity dependent!}$$

e.g. if $v = 10^3 \text{ km/s}$ and $\nu = 10^{10} \text{ Hz}$ then $b_{\max} \simeq \frac{10^6 \text{ m s}^{-1}}{4 \times 10^{10} \text{ s}^{-1}} \approx 2.5 \times 10^{-5} \text{ m}$

Thermal distribution $\rightarrow$ range of velocities

Order of magnitude $\frac{3}{2} kT = \frac{1}{2} m \langle v^2 \rangle$

$$\langle v \rangle \simeq \sqrt{3kT} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \text{ kg m}^2 \text{ s}^{-2} \text{ } ^\circ\text{K}^{-1} T}{9.11 \times 10^{-31} \text{ kg}}} = 0.68 \times 10^4 \sqrt{T} \text{ m/s}$$

For a minimum temperature of 10^4 K we get a typical velocity of 680 km/s .

At 10^{10} Hz we get: $b_{\max} \simeq \frac{v}{4\nu} \simeq \frac{6.8 \times 10^5 \text{ m/s}}{4 \times 10^{10} \text{ s}^{-1}} \simeq 1.7 \times 10^{-5} \text{ m}$

$$b_{\min(q)} \geq \frac{h}{2\pi m_e v} \approx \frac{6.626 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}}{2\pi 9.11 \times 10^{-31} \text{ kg} \times 6.8 \times 10^5 \text{ m s}^{-1}} \approx 0.17 \times 10^{-10} \text{ m}$$

$$b_{\min(c)} \geq \frac{2Ze^2}{m_e v^2} \dots$$

short range interaction

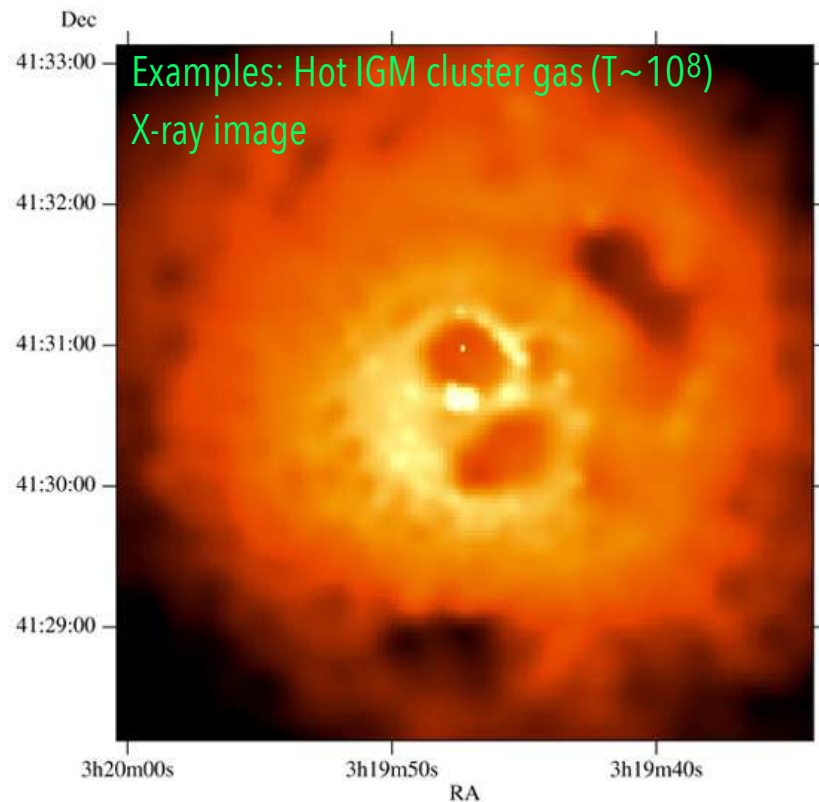
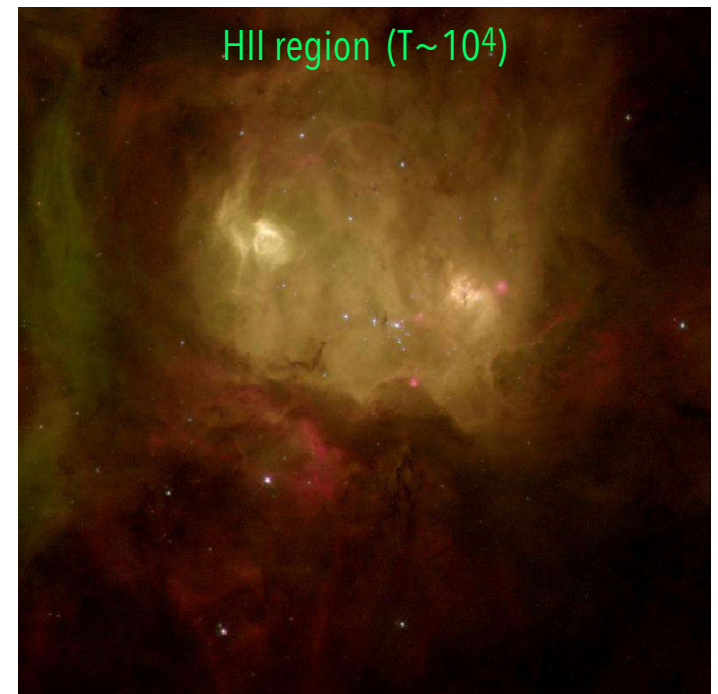
FYI

which [classical/quantum] limit does prevail?

$$\frac{b_{\text{min-q}}}{b_{\text{min-c}}} \approx \frac{h/(2\pi m_e v)}{2Ze^2/(m_e v^2)} = \frac{h}{4\pi Ze^2} c \frac{v}{c} \approx \frac{137}{Z} \frac{v}{c} > 1 \text{ when } v > 0.01 c$$

but $v = \sqrt{3kT/m_e}$ then

$$\frac{b_{\text{min-q}}}{b_{\text{min-c}}} \approx \frac{137}{Zc} \sqrt{\frac{3k}{m_e}} T^{1/2}$$

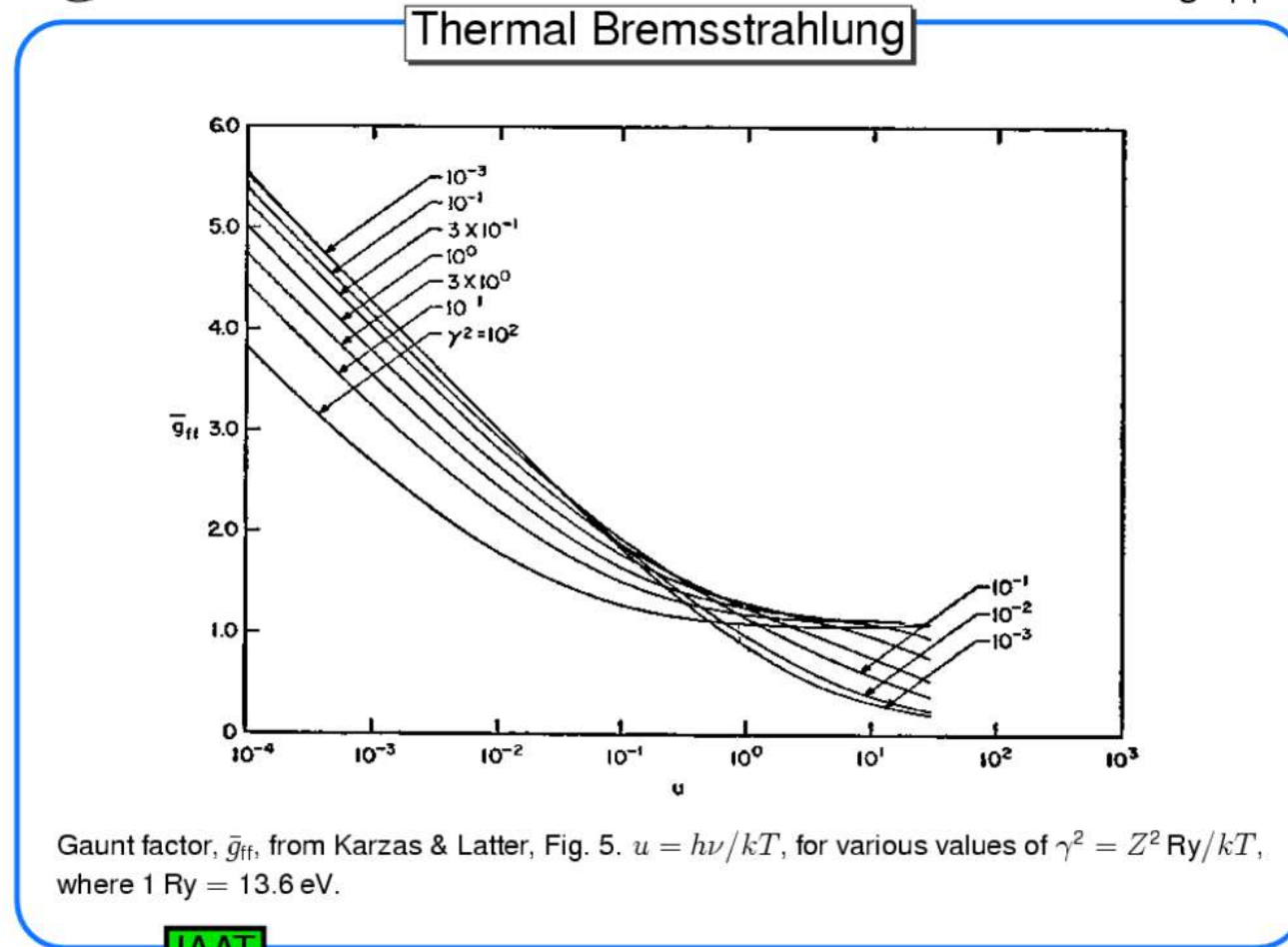


$$g_{\text{ff}}(\nu, T) = \frac{\sqrt{3}}{\pi} \ln \left(\frac{b_{\text{max}}}{b_{\text{min}}} \right)$$

The Gaunt Factor ranges from about 1 to about 10 in the radio domain



5-11



IAAT

Classical Treatment

8

Values around 10-15 in the radio domain, slightly higher than 1 at higher

bremsstrahlung – interlude: photon frequency .vs. temperature

Consider an electron moving at a speed of $v = 1000 \text{ km s}^{-1}$.

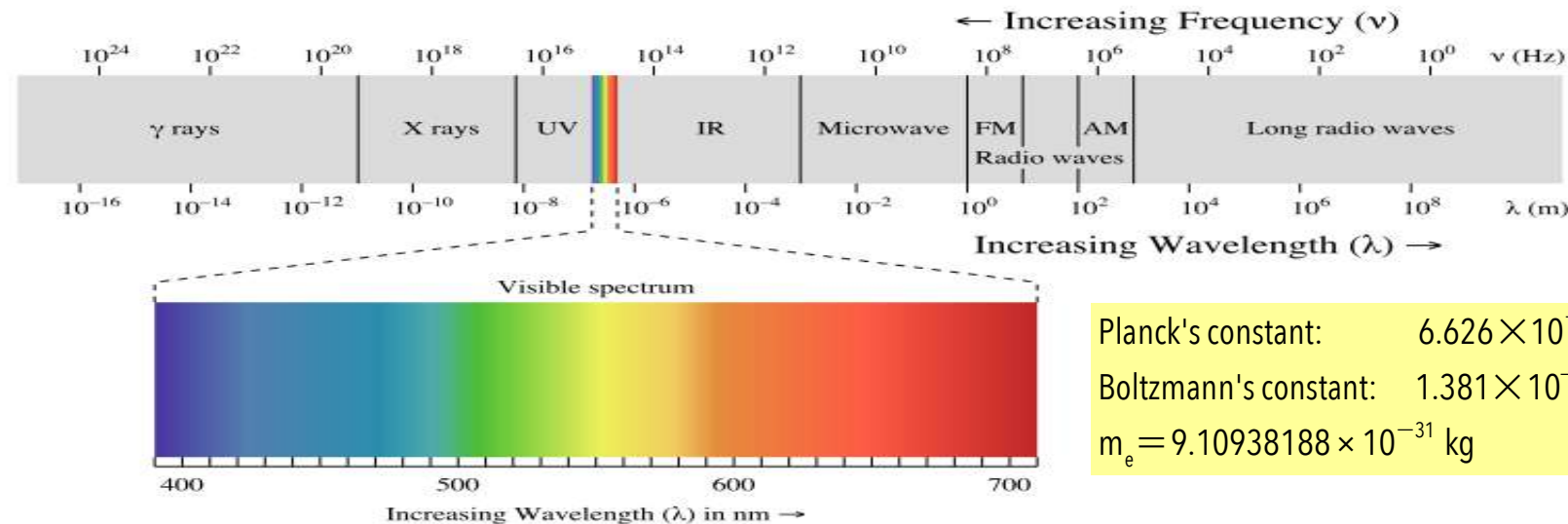
In case it would radiate all its kinetic energy in a single interaction

$$\Delta E = h\nu = \frac{1}{2}mv^2 = 0.5 \times 9.1 \times 10^{-31} \text{ kg} \times (10^6 \text{ m s}^{-1})^2 = 4.55 \times 10^{-19} \text{ J}$$

the frequency of the emitted radiation is $\nu = \frac{\Delta E}{h} = \frac{4.55 \times 10^{-19} \text{ kg m}^2 \text{ s}^{-2}}{6.626 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}} = 6.87 \times 10^{14} \text{ Hz}$

in case this electron belongs to a population of particles at thermal equilibrium for which $v = 1000 \text{ km s}^{-1}$ is the typical speed, determine which is the temperature of the plasma

$$T = \frac{\Delta E}{k} = \frac{4.55 \times 10^{-19} \text{ kg m}^2 \text{ s}^{-2}}{1.38 \times 10^{-23} \text{ kg m}^2 \text{ s}^{-2} \text{ K}^{-1}} = 3.30 \times 10^4 \text{ K}$$



Planck's constant: $6.626 \times 10^{-27} \text{ erg s}^{-1} = 6.626 \times 10^{-34} \text{ J s}^{-1}$

Boltzmann's constant: $1.381 \times 10^{-16} \text{ erg K}^{-1} = 1.381 \times 10^{-23} \text{ J K}^{-1}$

$m_e = 9.10938188 \times 10^{-31} \text{ kg}$

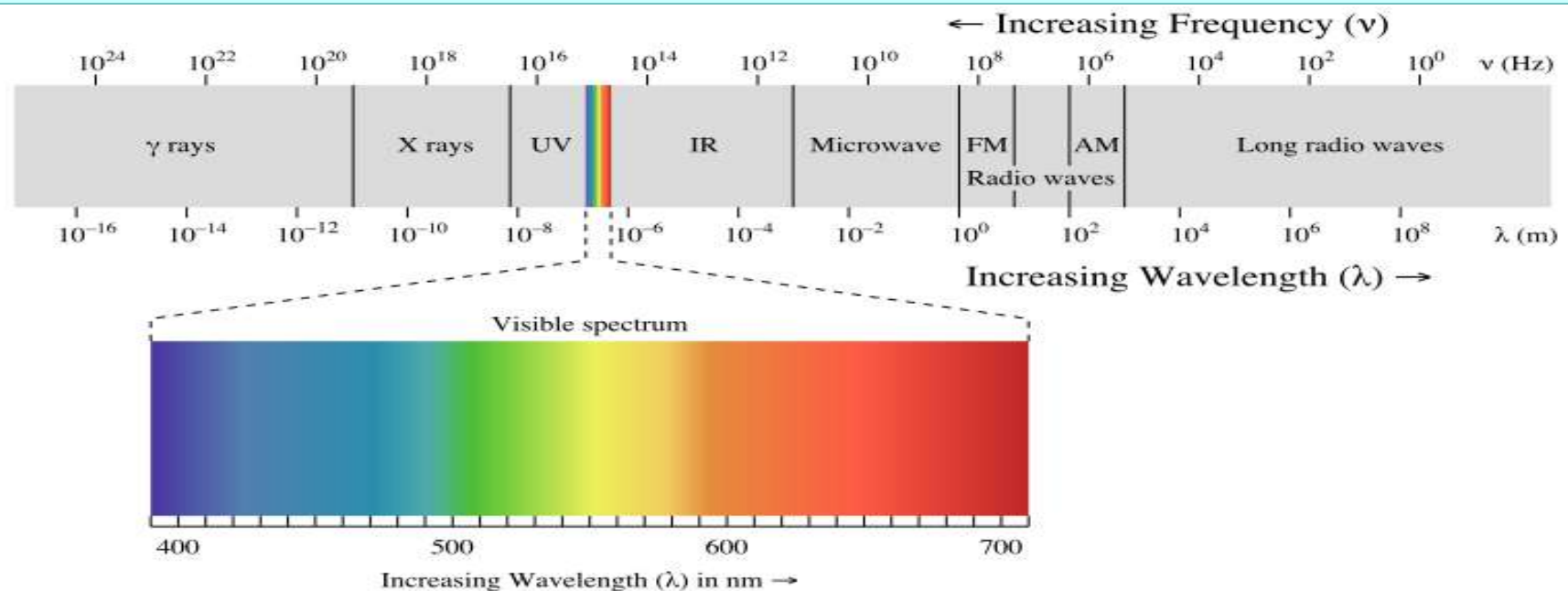
Bremsstrahlung – interlude (2): the cut-off frequency

The cut-off frequency in the thermal bremsstrahlung spectrum depends on the temperature only, and is conventionally set when the exponential equals $1/e$.

$$\frac{h\nu}{kT} = 1 \rightarrow \nu = \frac{k}{h} T = \frac{1.38 \times 10^{-23} \text{ kg m}^2 \text{ s}^{-2} \text{ K}^{-1}}{6.626 \times 10^{-34} \text{ kg m}^2 \text{ s}^{-1}} \times T = 2.08 \times 10^{10} \times T \text{ (Hz)}$$

It is interesting to determine the cut-off frequency for a warm plasma (HII region, $T \sim 10^4 \text{ K}$) and for a hot plasma (IGM in clusters of galaxies, $T \sim 10^8 \text{ K}$).

⇒ *In observations, the measure of the cut-off frequency is a way to determine the plasma temperature*



Total emissivity does not depend on frequency
(except the cut - off) and the spectrum is flat

$$J_{\text{br}}(\nu, T) = 6.8 \cdot 10^{-38} T^{-1/2} e^{-h\nu/kT} n_e n_z Z^2 g_{\text{ff}}(\nu, T) \quad \text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1}$$

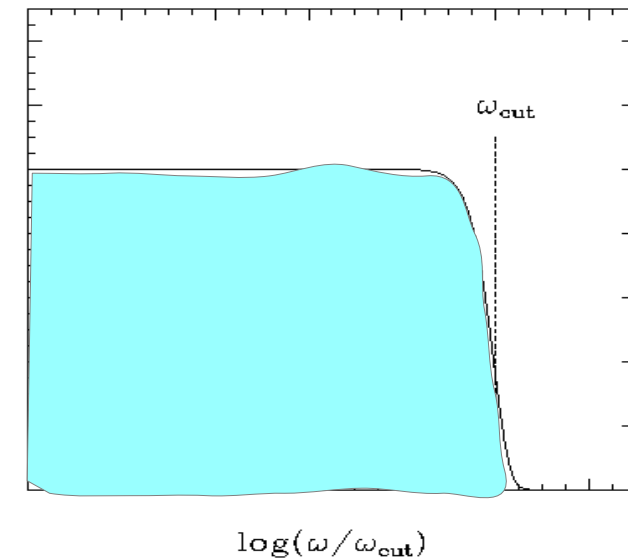
Total emitted energy per unit volume: must integrate
over velocity (energy) distribution ($0 \leq v < \infty$?) at a
given frequency ν , the velocity of the particle v must be as such

$$\frac{1}{2} m v^2 \geq h \nu \quad \rightarrow \quad v_{\text{min}} \geq \sqrt{\frac{2 h \nu}{m_e}}$$

and integrating over the whole spectrum:

$$J_{\text{br}}(T) = \frac{dE}{dt dV} \simeq 2.4 \cdot 10^{-27} T^{1/2} n_e n_z Z^2 \bar{g}_{\text{ff}}(T) \quad \text{erg s}^{-1} \text{cm}^{-3}$$

$\bar{g}_{\text{ff}}(T)$ average gaunt factor



Let's consider the total thermal energy of a plasma and the radiative losses:
the **cooling time** is

$$t_{\text{br}} = \frac{E_{\text{th}}^{\text{tot}}}{J_{\text{br}}(T)} = \frac{3/2(n_e + n_p)kT}{J_{\text{br}}(T)} \quad \text{we can assume} \quad n_e = n_p$$

$$\simeq \frac{3n_e kT}{2.4 \cdot 10^{-27} T^{1/2} n_e^2 \bar{g}_{\text{ff}}} = \frac{1.8 \cdot 10^{11}}{n_e \bar{g}_{\text{ff}}} T^{1/2} \text{ sec} \rightarrow = \frac{6 \cdot 10^3}{n_e \bar{g}_{\text{ff}}} T^{1/2} \text{ yr}$$

N.B. The "average" Gaunt Factor is about a factor of a few

Astrophysical examples:

HII regions: $n_e \sim 10^2 - 10^3 \text{ cm}^{-3}$; $T \sim 10^4 \text{ K}$

Galaxy clusters $n_e \sim 10^{-3} \text{ cm}^{-3}$; $T \sim 10^7 - 10^8 \text{ K}$

(thermal) bremsstrahlung is the main cooling process at temperatures above $T \sim 10^7 \text{ K}$

N.B.-2- all galaxy clusters have bremsstrahlung emission

A galactic HII region: **M42 (Orion nebula)**

Distance: 1.6 kly $\sim$ 0.5 kpc angular size $\sim 66'$ $\rightarrow 1.1^\circ$

$T = 10^4 \text{ } ^\circ\text{K}$, $S_{1.4\text{GHz}} = 410 \text{ Jy}$

Let's get some astrophysical information

$$\nu_{\text{cutoff}} \sim \frac{k}{h} T = 2.08 \cdot 10^{10} \left(\frac{T}{^\circ\text{K}} \right) \text{ Hz}$$

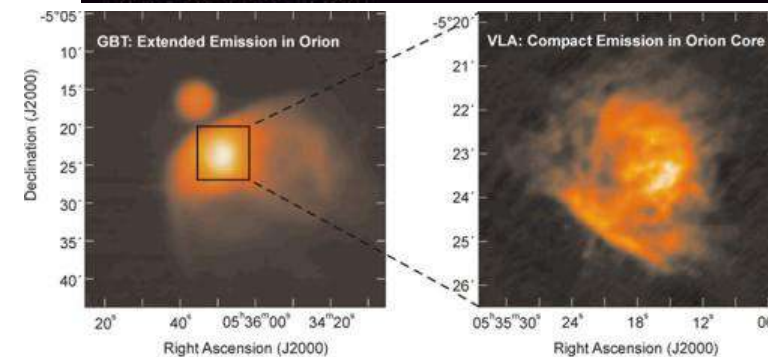
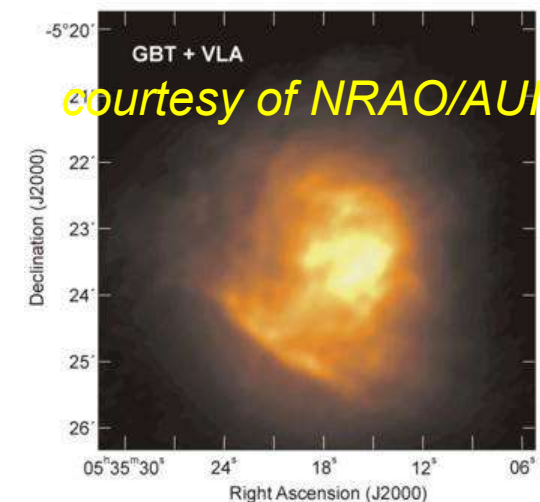
$$S = \frac{J_{\text{br}}(\nu, T) \cdot V}{4\pi D^2}$$

$$t_{\text{br}} = \frac{6 \cdot 10^3}{n_e \bar{g}_{\text{ff}}} T^{1/2} \text{ yr}$$

$$j_{\text{br}}(T) = 2.4 \cdot 10^{-27} T^{1/2} n_e^2 \bar{g}_{\text{ff}} \text{ erg s}^{-1} \text{ cm}^{-3}$$

$$L_{\text{br}} = j_{\text{br}} \cdot V \text{ erg s}^{-1}$$

Pay attention to the UNITS



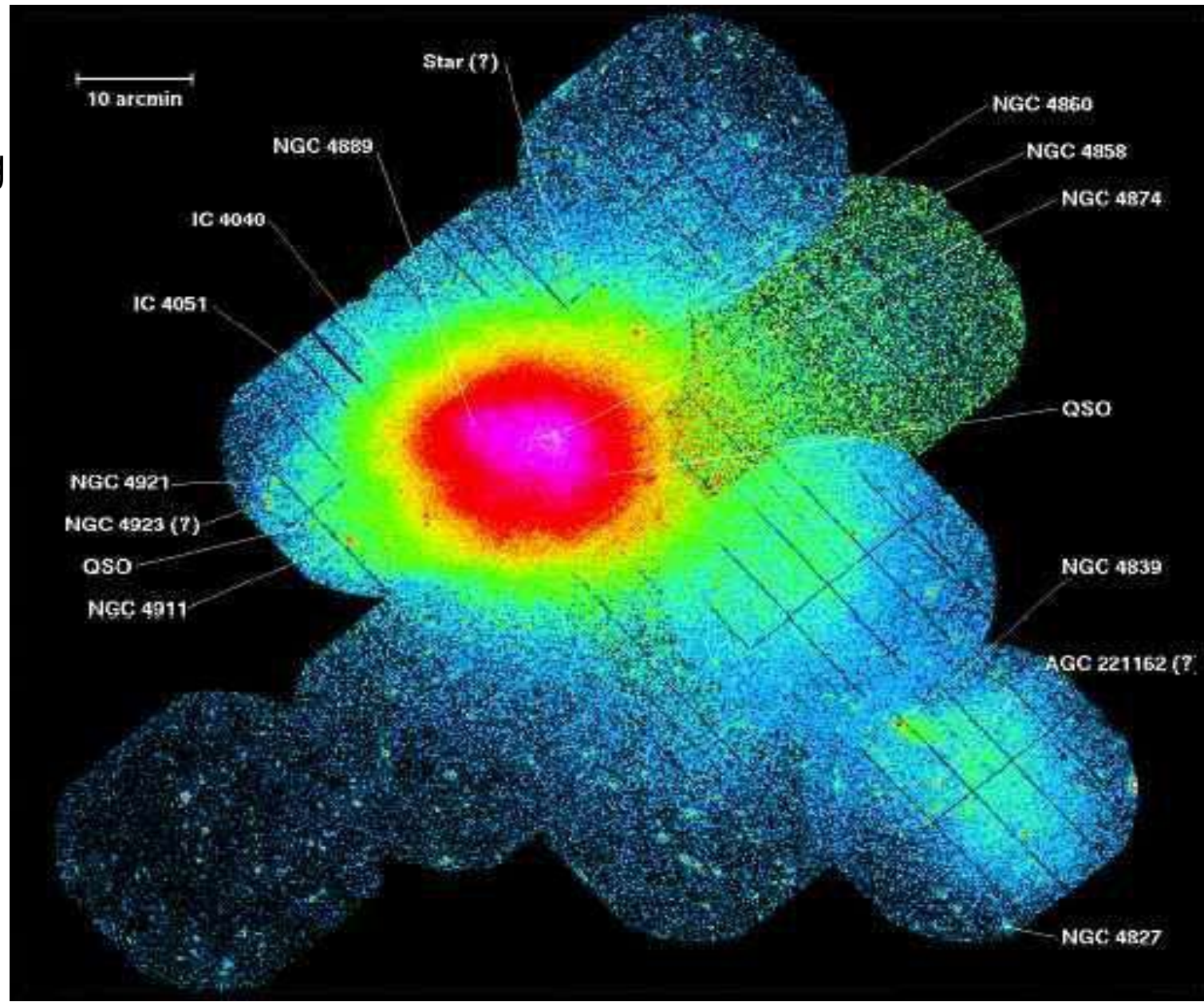
courtesy of NRAO/AUI

A cluster of galaxies:

Coma ($z=0.0232$), size ~ 1 Mpc

Typical values of IC hot gas have radiative cooling time exceeding 10 Gyr

All galaxy clusters are X-ray emitters



Thermal free-free absorption: energy gained by a free moving electron.

For a Maxwellian velocity (energy) distribution, i.e at thermal equilibrium, Kirchhoff's law holds

$$S(\nu) = \frac{j(\nu, T)}{\mu(\nu, T)} = B(\nu, T) = 2 \frac{h \nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

Optically thick regime =
Black Body photon distribution

$$j(\nu, T) = \mu(\nu, T) B(\nu, T) \quad [\text{Kirchhoff's law}]$$

For an isotropically emitting plasma cloud via thermal bremsstrahlung

$$\frac{dE}{dt dV d\nu} = 4\pi J_{br}(\nu, T)$$

Therefore we can get

$$\mu_{br}(\nu, T) = \frac{J_{br}(\nu, T)}{4\pi B(\nu, T)} = \frac{1}{4\pi} 6.8 \cdot 10^{-38} T^{-1/2} n_e n_z Z^2 \bar{g}_{ff}(\nu, T) \frac{c^2}{2h\nu^3} (1 - e^{-h\nu/kT})$$

$$\Rightarrow \mu_{br}(\nu, T) \approx T^{-1/2} n_e n_z \nu^{-3} (1 - e^{-h\nu/kT})$$

$\Rightarrow$ strong dependency on frequency

- at high frequencies $\mu(\nu, T)$ is negligible (ν^{-3} and $e^{-h\nu/kT}$)

- at low frequencies ($h\nu \ll kT$)

$$\begin{aligned}\mu_{\text{br}}(\nu, T) &\approx T^{-1/2} n_e n_z \nu^{-3} (1 - e^{-h\nu/kT}) \\ &\approx T^{-1/2} n_e n_z \nu^{-3} \left(1 - 1 + \frac{h\nu}{kT} \right) = n_e n_z T^{-3/2} \nu^{-2}\end{aligned}$$

$$\mu_{\text{br}}(\nu, T) = \frac{4e^6}{3m_e h c} \left(\frac{2\pi}{3km_e} \right)^{1/2} n_e n_z Z^2 T^{-3/2} \nu^{-2} \bar{g}_{\text{ff}}(T)$$

$$\bar{g}_{\text{ff}}(T) \approx 10 \quad \text{therefore}$$

$$\mu_{\text{br}}(\nu, T) \approx 0.018 T^{-3/2} \nu^{-2} n_e n_z Z^2 \bar{g}_{\text{ff}}(T) \simeq 0.2 T^{-3/2} \nu^{-2} n_e n_z Z^2$$

The total brightness from a free-free emitting cloud:

let's consider emission + absorption

$$B(\nu, T) = \frac{1}{4\pi} \frac{j_{\text{br}}(\nu, T)}{\mu_{\text{br}}(\nu, T)} (1 - e^{-\tau(\nu, T)}) = B_{\text{BB}}(\nu, T)(1 - e^{-\tau(\nu, T)})$$

according to radiative transfer

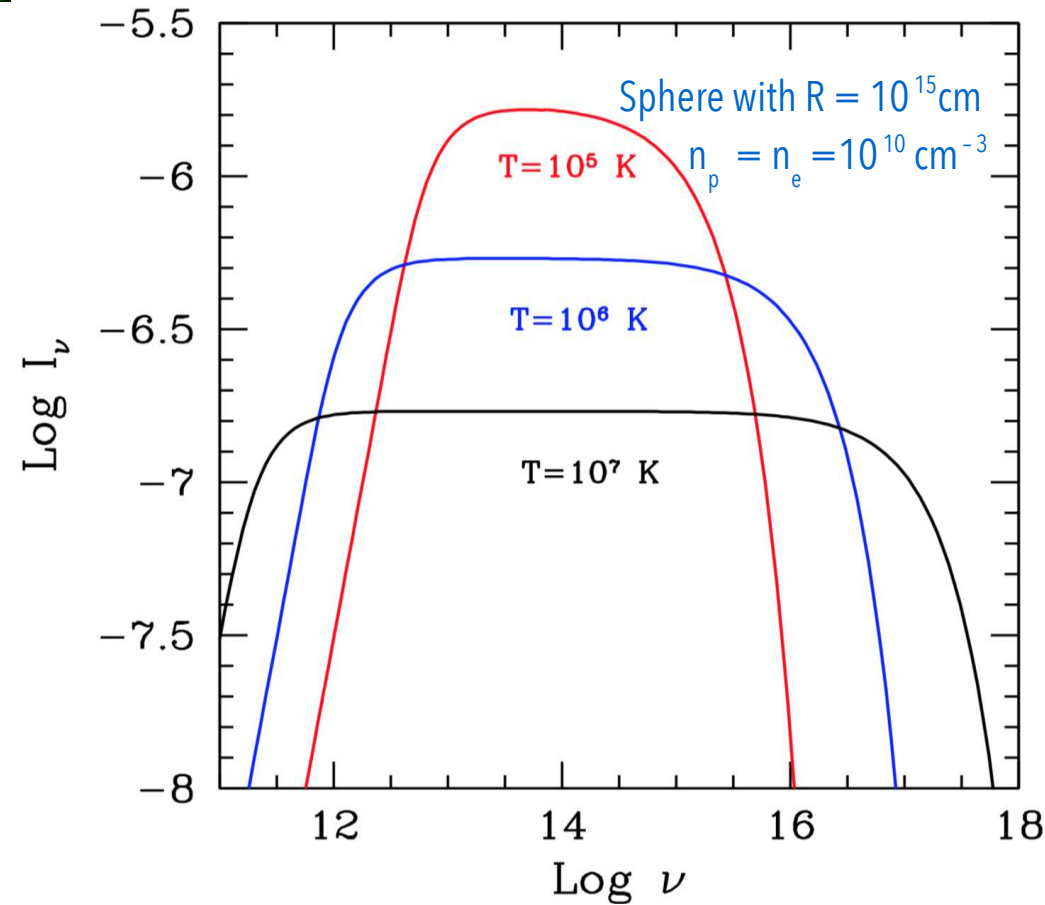
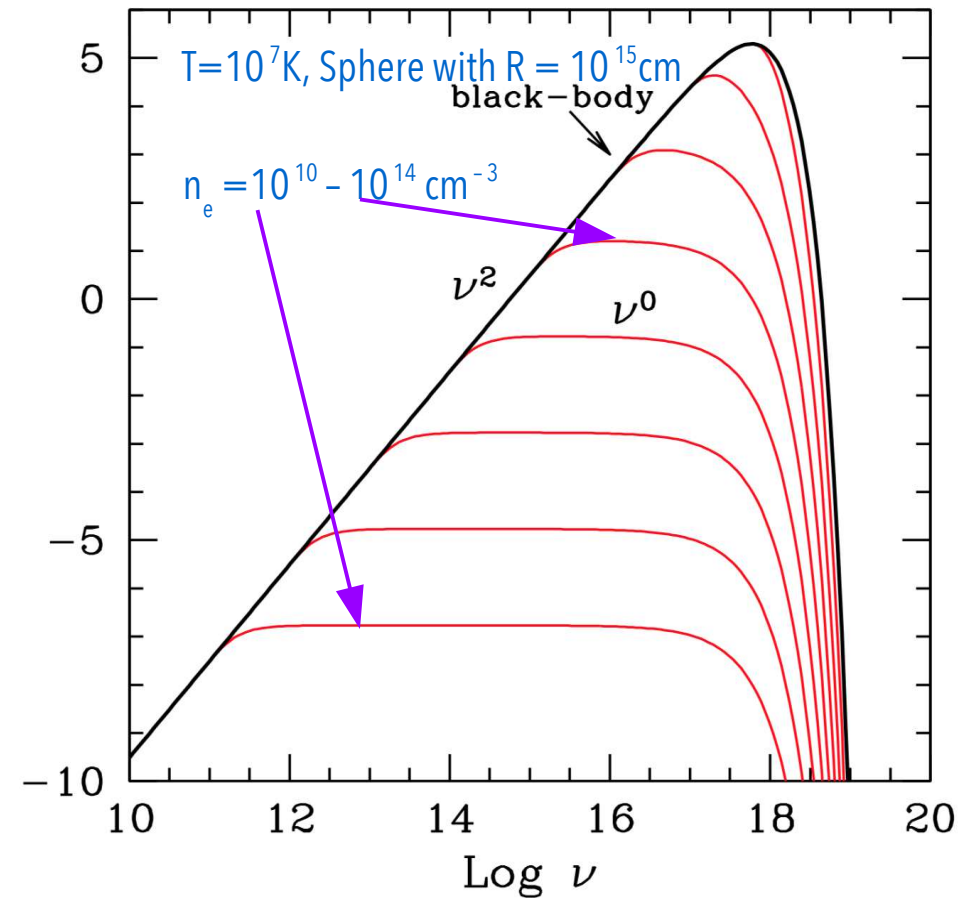
Therefore:

$$B(\nu, T) \approx \frac{T^{-1/2} n_e n_z Z^2 \bar{g}_{\text{ff}}(\nu, T)}{T^{-3/2} \nu^{-2} n_e n_z Z^2 \bar{g}_{\text{ff}}(\nu, T)} (1 - e^{-\tau(\nu, T)}) \approx T \nu^2 (1 - e^{-\tau(\nu, T)})$$

The opacity [$\tau(\nu, T) = \mu(\nu, T)l$; where l = mean free path] determines the spectral shape and the regime:

1. optically thick $\tau(\nu, T) \gg 1$

2. optically thin $\tau(\nu, T) \ll 1$



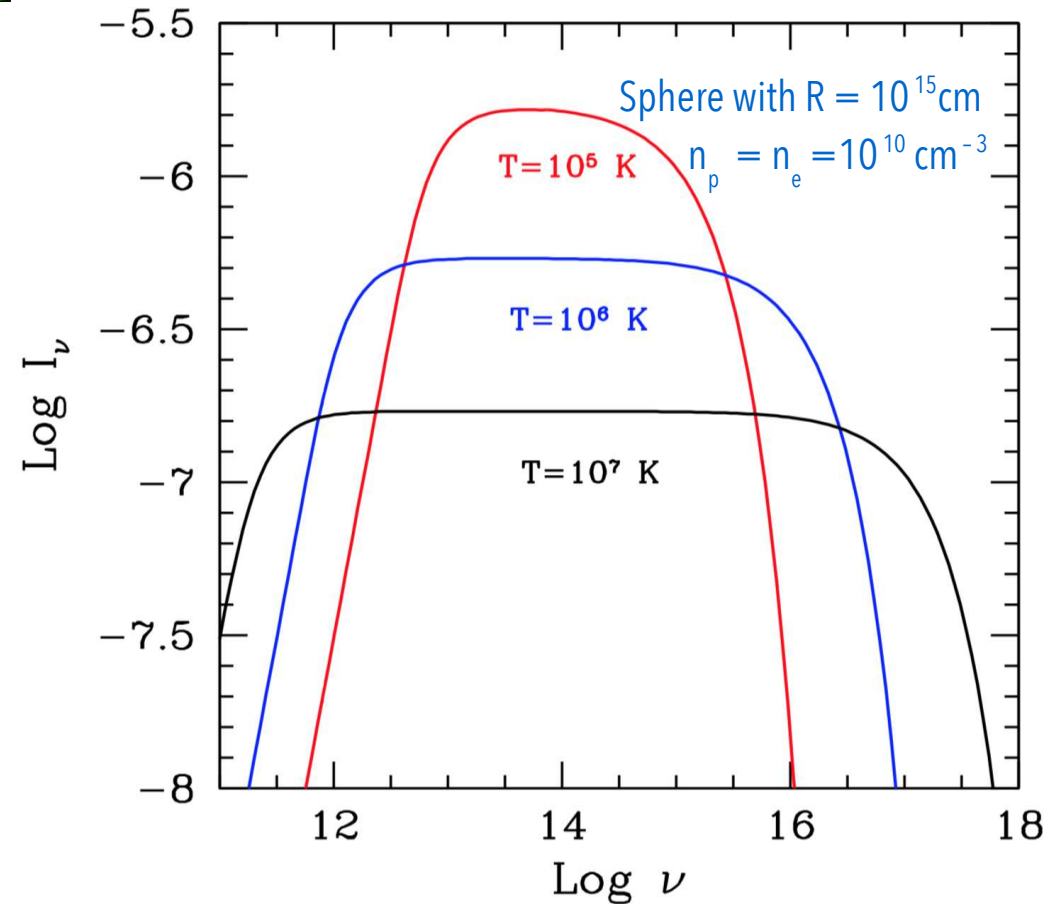
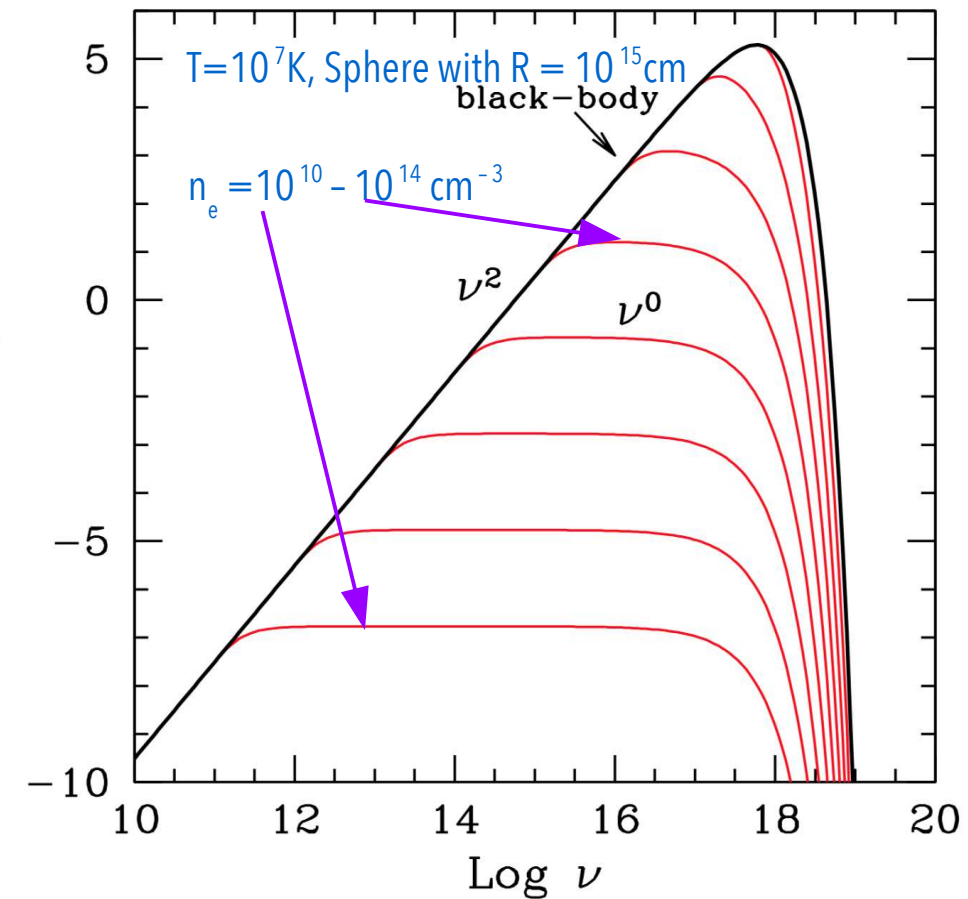
$$B(\nu, T) \approx T \nu^2 (1 - e^{-\tau(\nu, T)})$$

$$\tau \gg 1 \rightarrow (1 - e^{-\tau}) = \left(1 - \frac{1}{e^\tau}\right) \approx 1$$

$$\rightarrow B(\nu, T) \approx T \nu^2$$

$$\tau \ll 1 \rightarrow (1 - e^{-\tau}) = (1 - 1 + \tau - o(\tau^2)) \approx \tau = \mu l_0$$

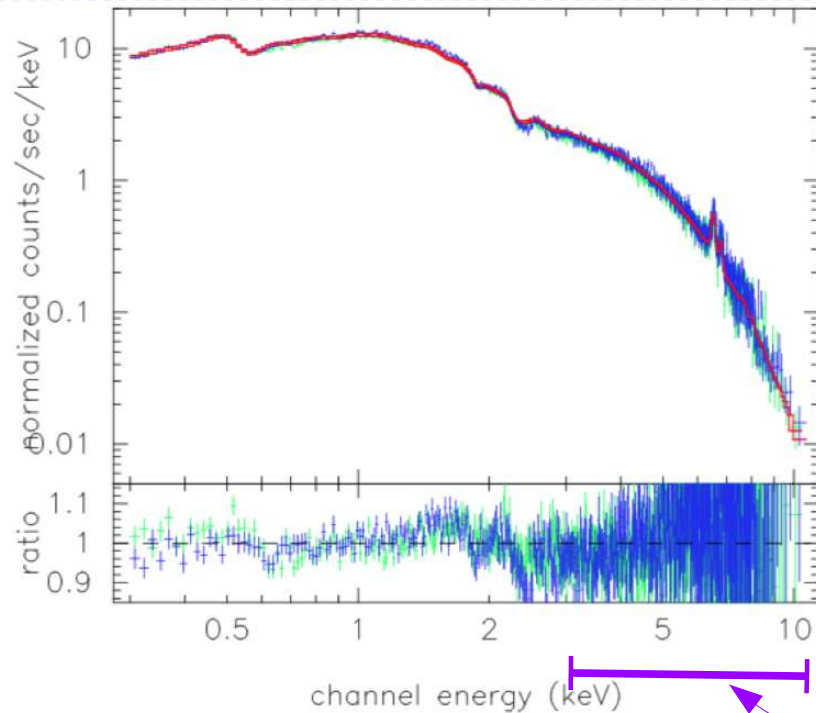
$$\text{but } \mu(\nu) \sim \nu^{-2} \rightarrow B(\nu, T) \approx T \nu^2 \tau \approx T \nu^0$$



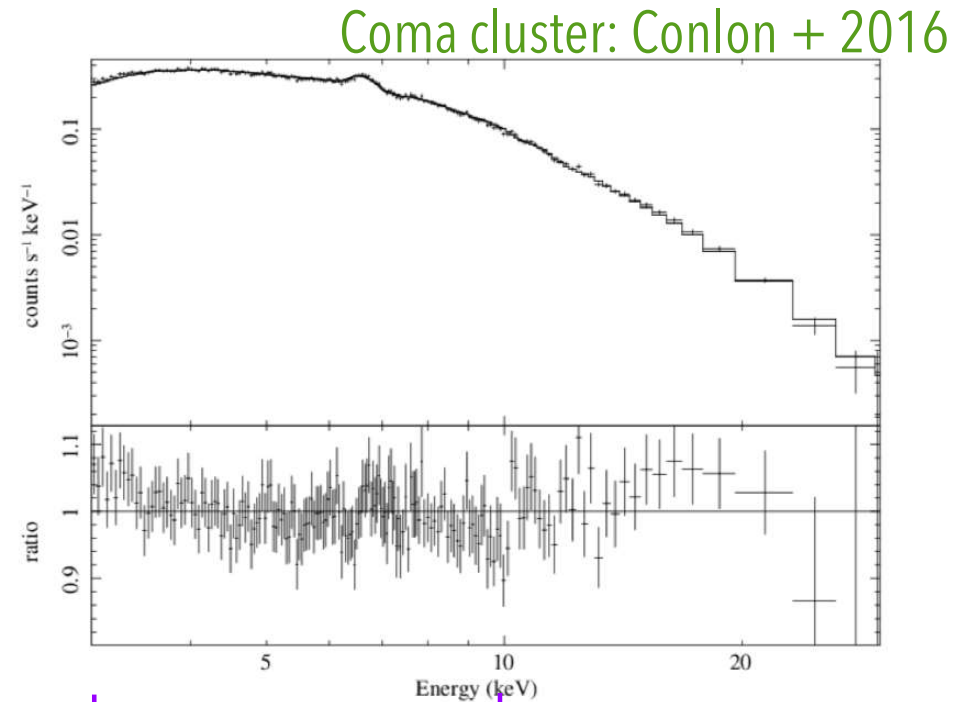
Fixed size and density ↑
 Emissivity decreases with temperature $[J_{\text{br}}(\nu, T) \approx T^{-1/2}]$
 The cut-off frequency increases linearly with T ...and
 Self-absorption becomes less effective $[\mu(\nu, T) \approx T^{-3/2}]$

Fixed size and temperature ↑
 Emissivity increases with density $[J_{\text{br}}(\nu, T) \approx n_e^2]$
 The cut-off frequency stays constant
 Self-absorption reaches higher ν $[\mu(\nu, T) \approx n_e^2]$

Bremsstrahlung : real X-ray spectra (3)



(a) XMM-Newton best fit flux model from Coma, within radius of ~ 300 kpc [18].



(b) NuSTAR spectrum and best-fit single temperature model for the central $12' \times 12'$ of the Coma cluster [19].

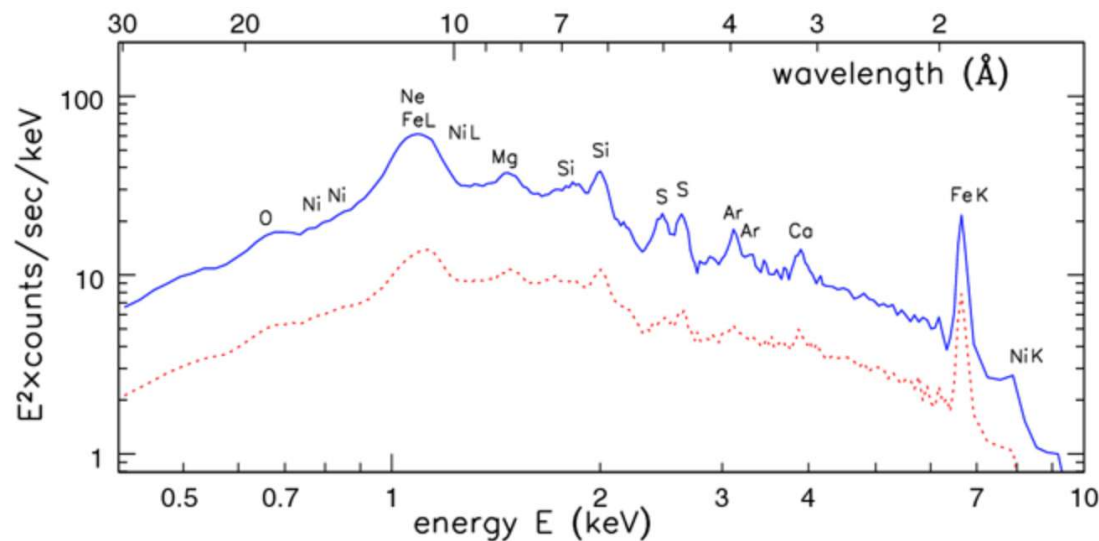


Fig 6.21 (K. Matsushita) 'Galaxies in the Universe' Sparke/Gallagher CUP 2007

The cut-off in these X-ray spectra is a measure of the plasma temperature

If distance is known, from the flux, we can derive the density (n_e)

Minor remark: the Gaunt Factor is slightly different wrt the case of "thermal br."

The emissivity for relativistic bremsstrahlung as a function of v for a given velocity v is:

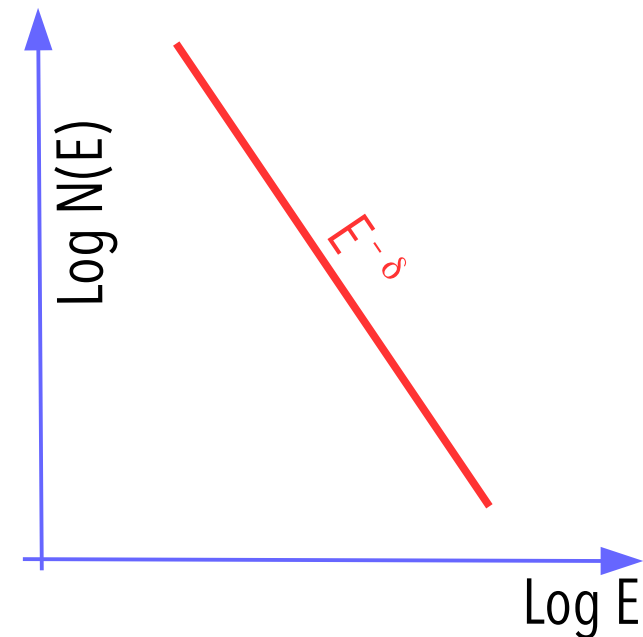
$$j_{\text{rel,br}}(v, \nu) = \frac{32\pi}{3} \frac{e^6}{m_e^2 c^3} \frac{1}{v} n_e n_z Z^2 \ln\left(\frac{183}{Z^{1/3}}\right)$$

➡ N.B. T is not a relevant concept any more, v must be used

The typical velocities of the particles are now relativistic and the energy (velocity) distribution is described by a power law:

➡
$$n_e(E) \approx n_{e,0} E^{-\delta}$$

Integrating over velocities (energies)



$$j_{\text{rel,br}}(\nu) \approx \int_{h\nu}^{\infty} n_e(E) n_z Z^2 dE \approx \int_{h\nu}^{\infty} E^{-\delta} dE \simeq \frac{E^{-\delta+1}}{1-\delta} \approx \nu^{-\delta+1}$$

➡ The emitted spectrum is a power-law !

solar wind:

$$n_e \sim 4 - 7 \text{ cm}^{-3}$$

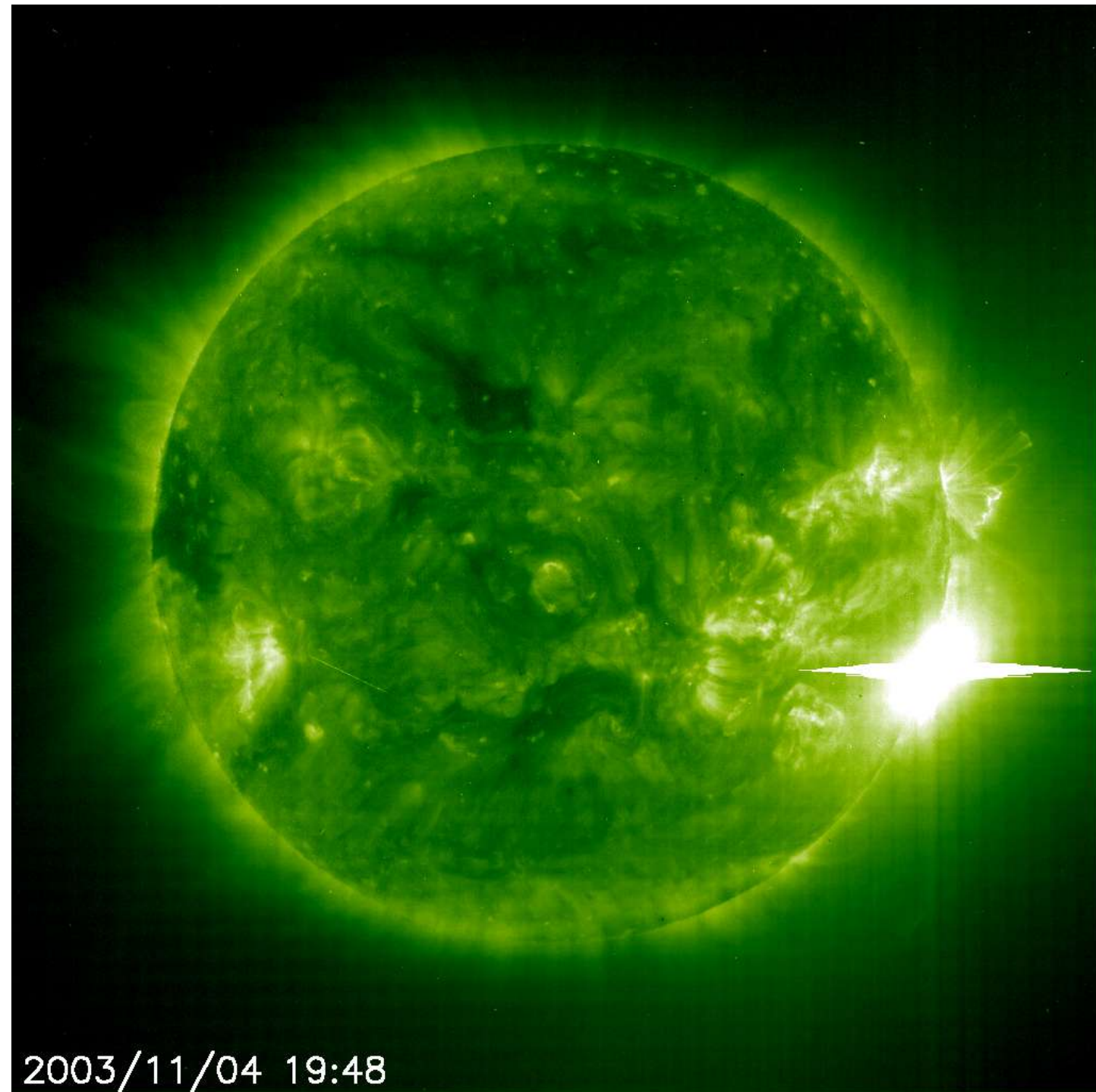
$$v_{\text{swind}} \sim 300 - 900 \text{ km s}^{-1}$$

$$T \sim 150\,000 \text{ K}$$

(not consistent with v !)

flares:

relativistic particles
with power-law energy
distribution





Final remarks on bremsstrahlung emission:

a photon is emitted when a moving electron deeply penetrates the Coulomb field of positive ion and then is decelerated

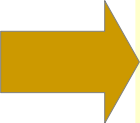
the energy of the photon can never exceed the kinetic energy of the electron

generally the "classical" description of the phenomenon is adequate, the quantum mechanics requires an appropriate Gaunt Factor only

the individual interaction produces polarized radiation (fixed direction of E and B fields of the e-m wave wrt the acceleration)

all the interactions produced in a hot plasma originate unpolarized emission (random orientation of the plane of interaction)

we have either "thermal" or "relativistic" bremsstrahlung, depending on the population (velocity/energy distribution) of the electrons.



Annoying question.... Given that only electrons radiate, they can cool, while positive ions do not... therefore, what happens to the thermal equilibrium?

TEST: Let's consider an HII region around an O-type star. The spherical volume (radius 1 pc) is homogeneously filled by a plasma of pure H, with $T = 1.69 \times 10^4 \text{ K}$ and $n_e = 10 \text{ cm}^{-3}$. Compute both the specific and bolometric luminosities (per unit volume). If the HII region is at a distance of 250 pc, which is the predicted flux density measured by a radiotelescope at 10 GHz (assume optically thin emission) In case the O star is removed from the system, how long the bremsstrahlung emission would last?

Total emissivity (per unit volume)

$$J_{\text{br}}(\nu, T) = 6.8 \cdot 10^{-38} T^{-1/2} e^{-h\nu/kT} n_e n_z Z^2 g_{\text{ff}}(\nu, T) \quad \text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1}$$

Total emitted energy per unit volume

$$J_{\text{br}}(T) = \frac{dW}{dt dV} \simeq 2.4 \cdot 10^{-27} T^{1/2} n_e n_z Z^2 \bar{g}_{\text{ff}}(T) \quad \text{erg s}^{-1} \text{cm}^{-3} \quad \bar{g}_{\text{ff}}(T) \quad \text{average gaunt factor}$$

Cooling Time

$$t_{\text{br}} = \frac{1.8 \cdot 10^{11}}{n_e \bar{g}_{\text{ff}}} T^{1/2} \quad \text{sec} \quad \rightarrow \quad = \frac{6 \cdot 10^3}{n_e \bar{g}_{\text{ff}}} T^{1/2} \quad \text{yr}$$

The "Cone Nebula" (NGC2264) is an HII region (an amateur image is shown below

It is at a distance of 800 pc and has an angular size of 3 deg.

Assuming that the gas is made of pure H, with $T = 10^4 \text{ K}$, and $n_e = 10^2 \text{ cm}^{-3}$, estimate the flux density at 30 GHz, assuming it is optically thin.



NGC2264 – Cone Nebula

Total emissivity (per unit volume)

$$J_{\text{br}}(\nu, T) = 6.8 \cdot 10^{-38} T^{-1/2} e^{-h\nu/kT} n_e n_z Z^2 g_{\text{ff}}(\nu, T) \quad \text{erg s}^{-1} \text{cm}^{-3} \text{Hz}^{-1}$$



Non-thermal Synchrotron Radiation on textbooks

G. Ghisellini: "Radiative Processes in High Energy Astrophysics" § 4

M. Longair: "High Energy Astrophysics" § 8

G. B. Rybicki & A.P. Lightman: "Radiative Processes in Astrophysics" § 6 (+4.7 & 4.8)

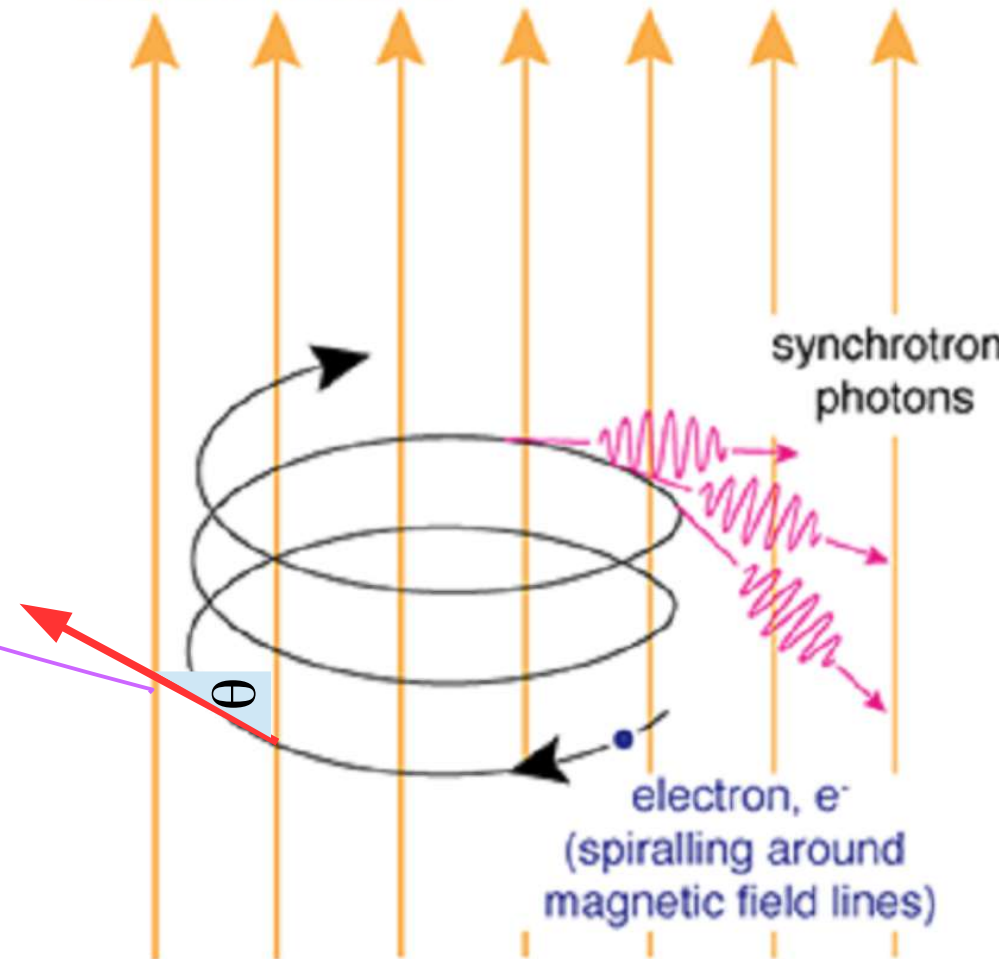
A charge q is deflected by a **uniform** magnetic field B according to:

$$\frac{d\vec{p}}{dt} = \frac{q}{c} \vec{v} \times \vec{H}$$

$$\rightarrow v_{\parallel} = v \cos \theta \quad v_{\perp} = v \sin \theta$$

θ is the "pitch angle" between the magnetic field and the velocity of the charge

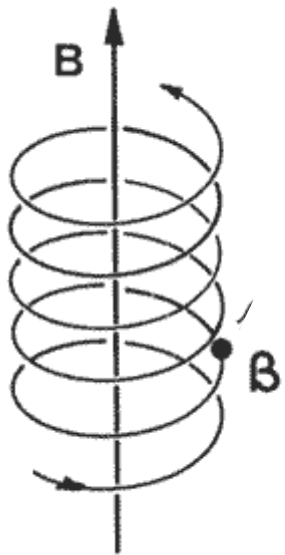
magnetic field lines



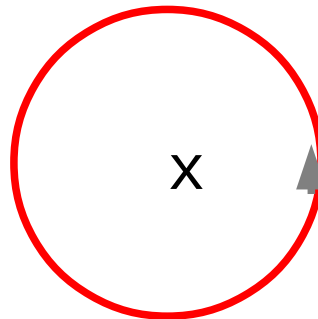
$$\frac{d(mv_{\parallel})}{dt} = 0 \quad \rightarrow \quad v_{\parallel} \text{ is constant} \quad \rightarrow \quad \text{The pitch angle remains constant}$$

$$\frac{d(mv_{\perp})}{dt} = \frac{q}{c} v_{\perp} \times H$$

In the direction perpendicular to the B field we have a circular motion
and in the non-relativistic case we have the well known **cyclotron** relations:



Side view



Top

$$r_L = \frac{mc}{qH} v_{\perp}$$

$$T_L = \frac{2\pi r_L}{v_{\perp}} = \frac{2\pi mc}{qH}$$

$$\omega_L = \frac{2\pi}{T_L} = \frac{qH}{mc}$$

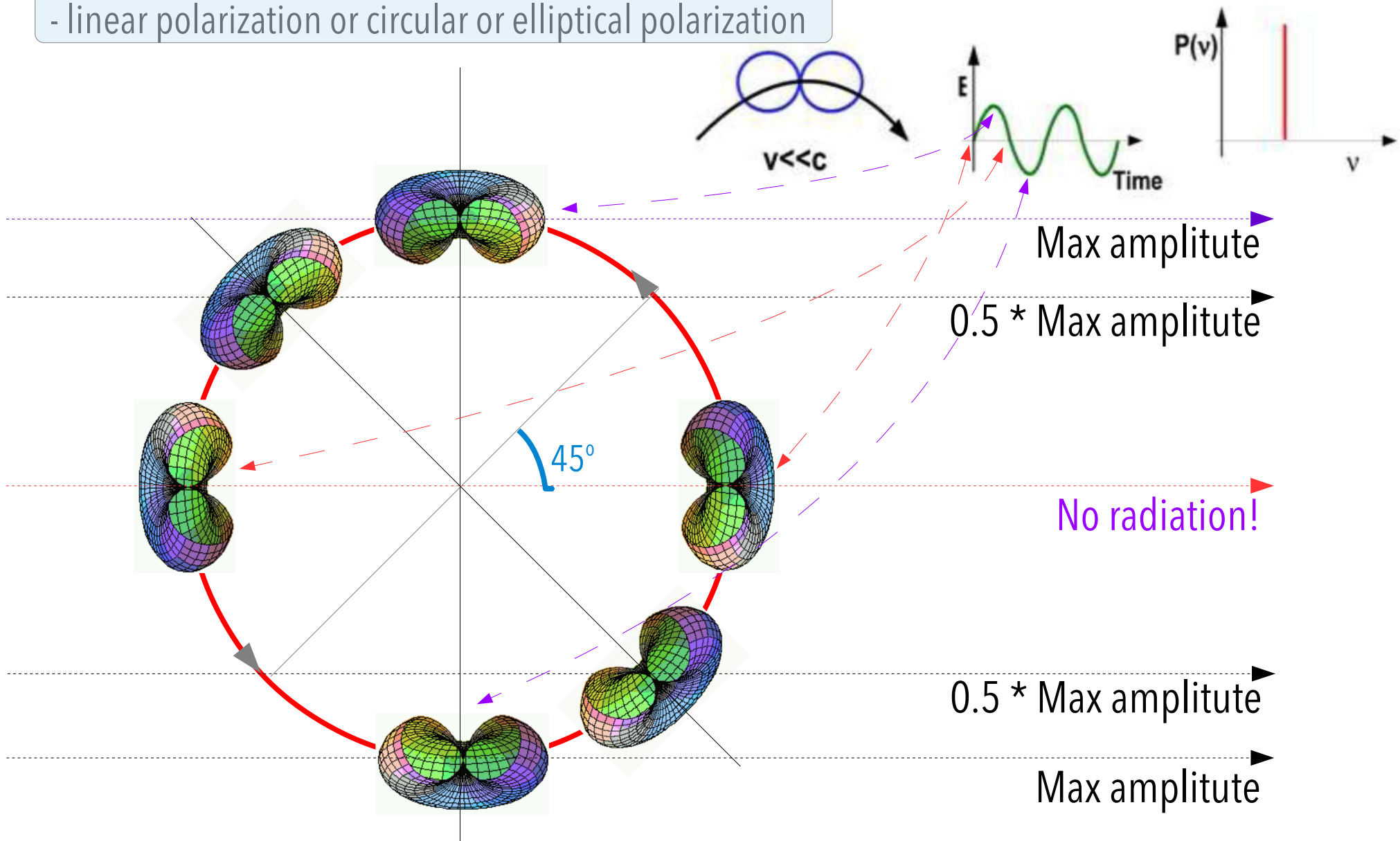
[c.g.s. units]

Cyclotron radiation

"A charge moving in a magnetic field is accelerated and then radiates [CYCLOTRON]"

- dipole in the plane of the circular orbit (dipole angular distribution)

- linear polarization or circular or elliptical polarization





Lorentz Force:
$$\frac{d\vec{p}}{dt} = m\vec{a} = \frac{q}{c} \vec{v} \times \vec{H}$$

in modulus
$$ma = \frac{q}{c} v H \sin \theta = q\beta H \sin \theta \rightarrow a = \frac{q}{m} \beta H \sin \theta$$

therefore in the Larmor's formula
$$\left(- \right) \frac{dE}{dt} = \frac{2}{3} \frac{q^4}{m^2 c^3} \beta^2 H^2 \sin^2 \theta$$

at the frequency
$$\nu_L = \frac{\omega_L}{2\pi} = \frac{qH}{2\pi mc}$$

If we consider an electron, its gyration frequency is:
$$\frac{\nu_L}{(\text{MHz})} \simeq 2.5 \frac{H}{\text{Gauss}}$$

N.B. In general, the magnetic fields are weak (μG), except in very particular stars, and collapsed bodies. $\rightarrow$ the emitted radiation happens at low energies

Example of a collapsed body: 34keV feature in Her-X1 (see Longair),

The relativistic case is dealt with by considering the appropriate expression of the particle mass:

$$m = \frac{m_0}{\sqrt{1-\beta^2}} = m_0 \cdot \gamma$$

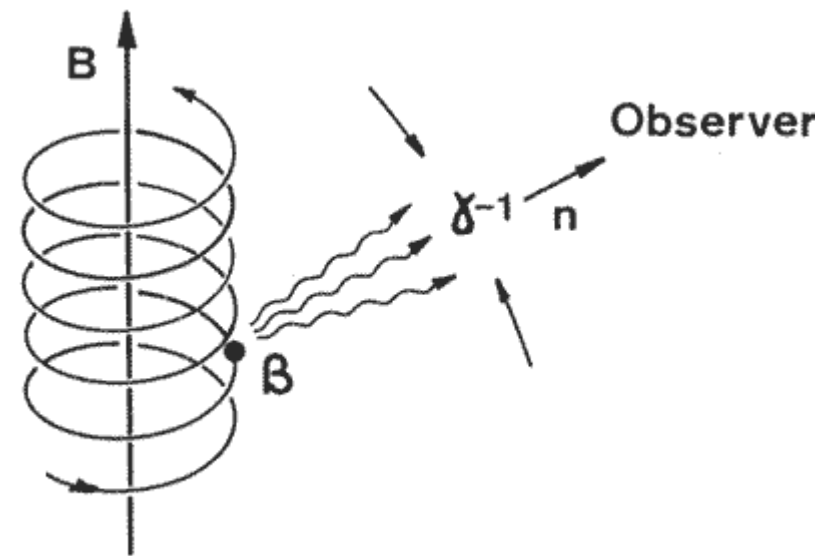
$$r_{L,rel} = \frac{\gamma m_0 c}{qH} v_{\perp} = \gamma r_L$$

$$T_{L,rel} = \frac{2\pi r_{L,rel}}{v_{\perp}} = \frac{2\pi m_0 c \gamma}{qH} = \gamma T_L$$

$$\omega_{L,rel} = \frac{2\pi}{T_{L,rel}} = \frac{qH}{\gamma m_0 c} = \frac{\omega_L}{\gamma}$$

if we consider $v \sim c$, we have:

$$r_{L,rel} \simeq \frac{\gamma m_0 c^2}{qH} = \frac{\varepsilon}{qH}$$



N.B. "classical" physics is ok, since for typical B fields strengths, $\hbar/p \ll r_{L,rel}$



[(ULTRA) - Relativistic] Charge ($\gamma \gg 1$) moving in a uniform magnetic field

Recall Larmor's Formula in the invariant form (scalars are not affected)

$$d\tau = \frac{dt}{\gamma}$$

$$\dot{\vec{p}} = [\vec{p}, (i/c)\vec{E}]$$

$$E^2 = p^2 c^2 + m^2 c^4$$

$$P = \frac{dE}{dt} = \frac{2q^2}{3m^2 c^3} \left[\frac{dp_i}{d\tau} \frac{dp_i}{d\tau} \right]$$

$$\left[\frac{dp_i}{d\tau} \frac{dp_i}{d\tau} \right] = \left(\frac{d\vec{v}}{d\tau} \right)^2 - \frac{1}{c^2} \left(\frac{dE}{d\tau} \right)^2 = \left(\frac{d\vec{v}}{d\tau} \right)^2 - \beta^2 \left(\frac{dp}{d\tau} \right)^2$$

FYI

Already seen earlier

$$P_{\text{rel}} = \frac{dE}{dt} = \frac{2q^2}{3m^2 c^3} \gamma^2 \left[\left(\frac{d\vec{v}}{d\tau} \right)^2 - \beta^2 \left(\frac{dp}{d\tau} \right)^2 \right]$$

Two options:



Classical, non relativistic, Larmor case $\rightarrow$ linear acceleration (no a_c)

$$\left| \frac{d\vec{p}}{d\tau} \right| \simeq \frac{dp}{d\tau}$$

$$P = \frac{dE}{dt} = \frac{2q^2}{3m^2c^3} \frac{1}{\gamma^2} \left(\frac{dp}{d\tau} \right)^2$$

$\rightarrow$ Centripetal acceleration $\left| \frac{d\vec{p}}{d\tau} \right| \gg \beta \left(\frac{dp}{d\tau} \right) = \frac{1}{c} \left(\frac{dE}{d\tau} \right)$

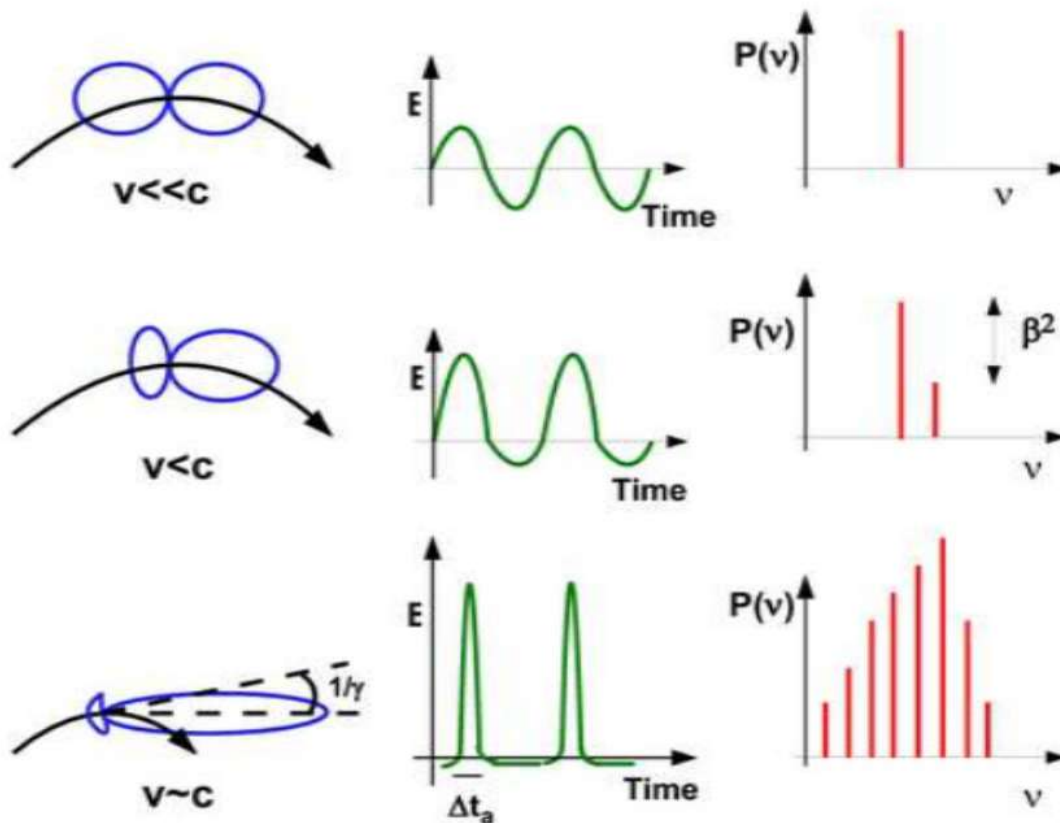
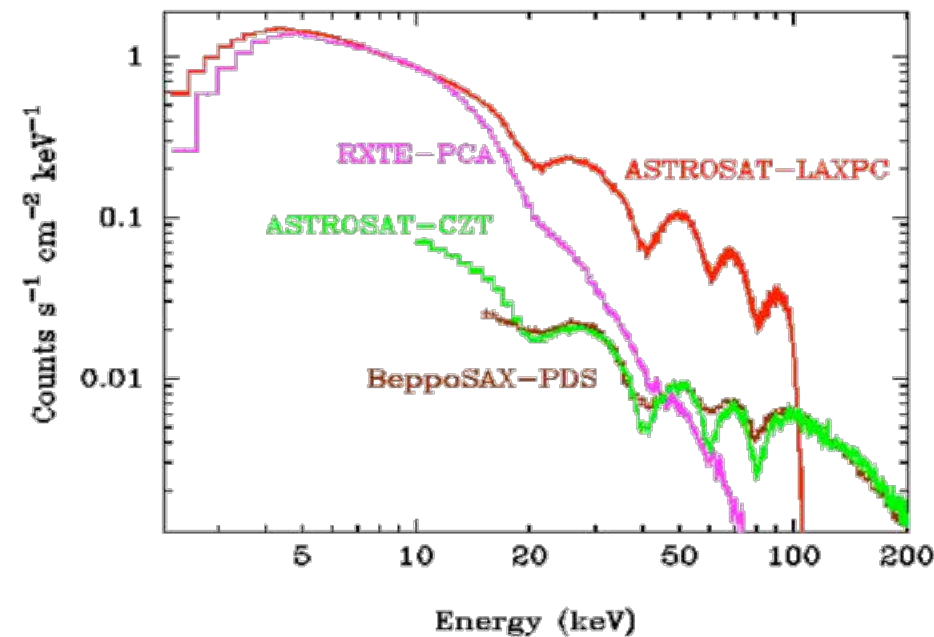
$$P = \frac{dE}{dt} = \frac{2q^2}{3m^2c^3} \left(\frac{d\vec{p}}{d\tau} \right)^2 = \frac{2q^2}{3m^2c^3} \gamma^2 \left(\frac{d\vec{p}}{dt} \right)^2$$

Relevant in relativistic cyclotron and synchrotron emission!

The energy is radiated in various harmonics of the gyration frequency:

$$\nu_k = k \nu_{\text{rel}} \left(1 - \frac{v_{\parallel} \cos \theta}{c} \right) \quad k=1,2,3,4,\dots n \quad \text{Doppler shift along the LoS}$$

The energy in the various harmonics follows: $\left[\frac{dE}{dt} \right]_{l+1} \approx \beta^2 \left[\frac{dE}{dt} \right]_l$



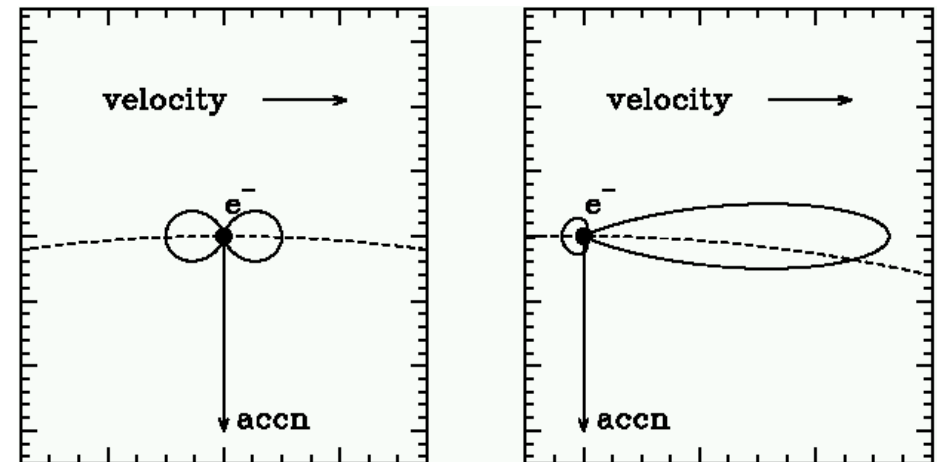


The energy in the various harmonics follows:

$$\left[\frac{dE}{dt} \right]_{n+1} \approx \beta^2 \left[\frac{dE}{dt} \right]_n$$

As a result of a relativistic effect on electron motion and its emitted radiation...

(Radiation) fields on the approaching side are **amplified**,
while they are attenuated on the receding side.



The effective field can therefore be represented as the superposition of a number of harmonics.

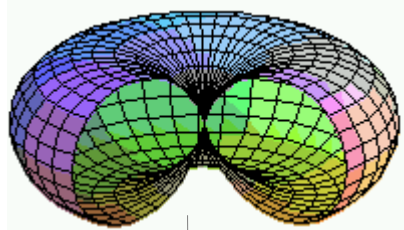
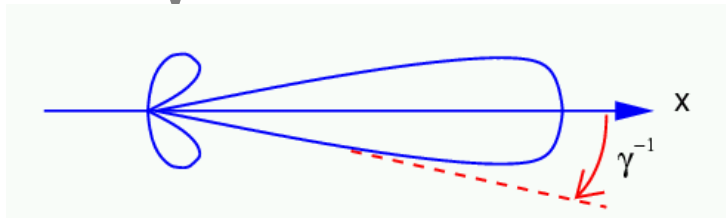
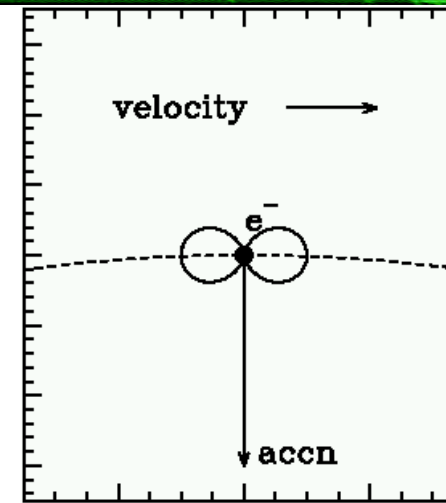
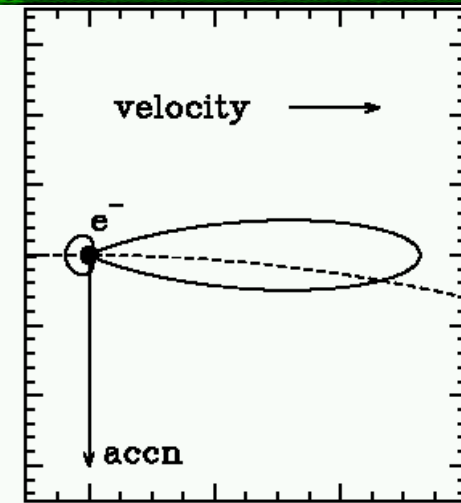
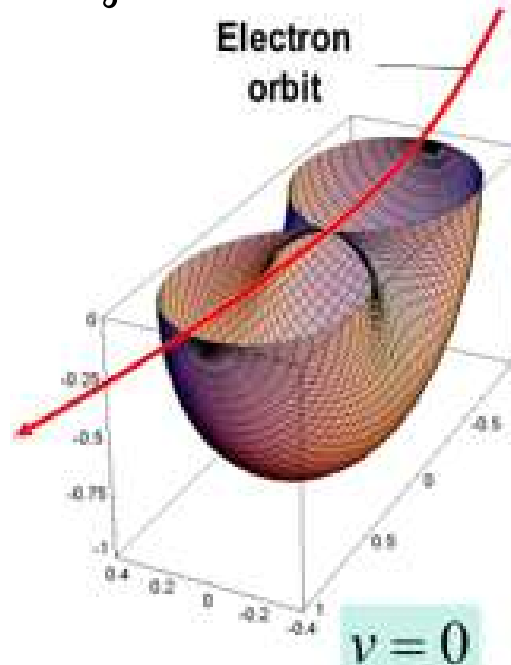
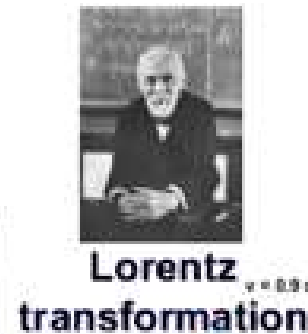
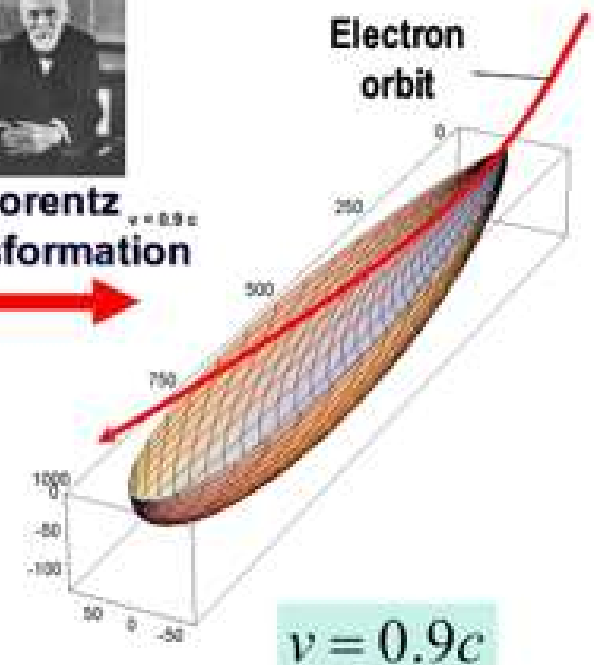
$$\tan(\alpha) = \frac{\sin(\alpha')(\sqrt{1-\beta^2})}{\cos \alpha' + \beta}$$

$$\alpha' = 90^\circ \rightarrow \tan(\alpha) \approx \sqrt{1-\beta^2} \approx \frac{1}{\gamma}$$

⇒ Amplification along the velocity direction

⇒ (relativistic beaming)

Half of the radiation is within a cone $\frac{2}{\gamma}$ wide





Synchrotron radiation

It is generated by ultra-relativistic electrons, for which the curvature of the trajectory plays the major role in determining the emitted power (single electron with its γ in a H field)

$$-\frac{dE}{dt} = \frac{2q^4}{3m^2c^3} \beta^2 \gamma^2 H^2 \sin^2 \theta$$

*H energy
density U_H*

if $\beta \simeq 1$ and $\epsilon = m_0 c^2 \gamma$ then it can be rewritten as

$$-\frac{dE}{dt} = \frac{2q^4}{3m_0^4c^7} \epsilon^2 H^2 \sin^2 \theta = 2c\sigma_T \gamma^2 \frac{H^2}{8\pi} \sin^2 \theta \simeq 1.62 \cdot 10^{-15} \gamma^2 H^2 \sin^2 \theta \text{ erg s}^{-1}$$

and σ_T is known as electron Thomson cross section defined as

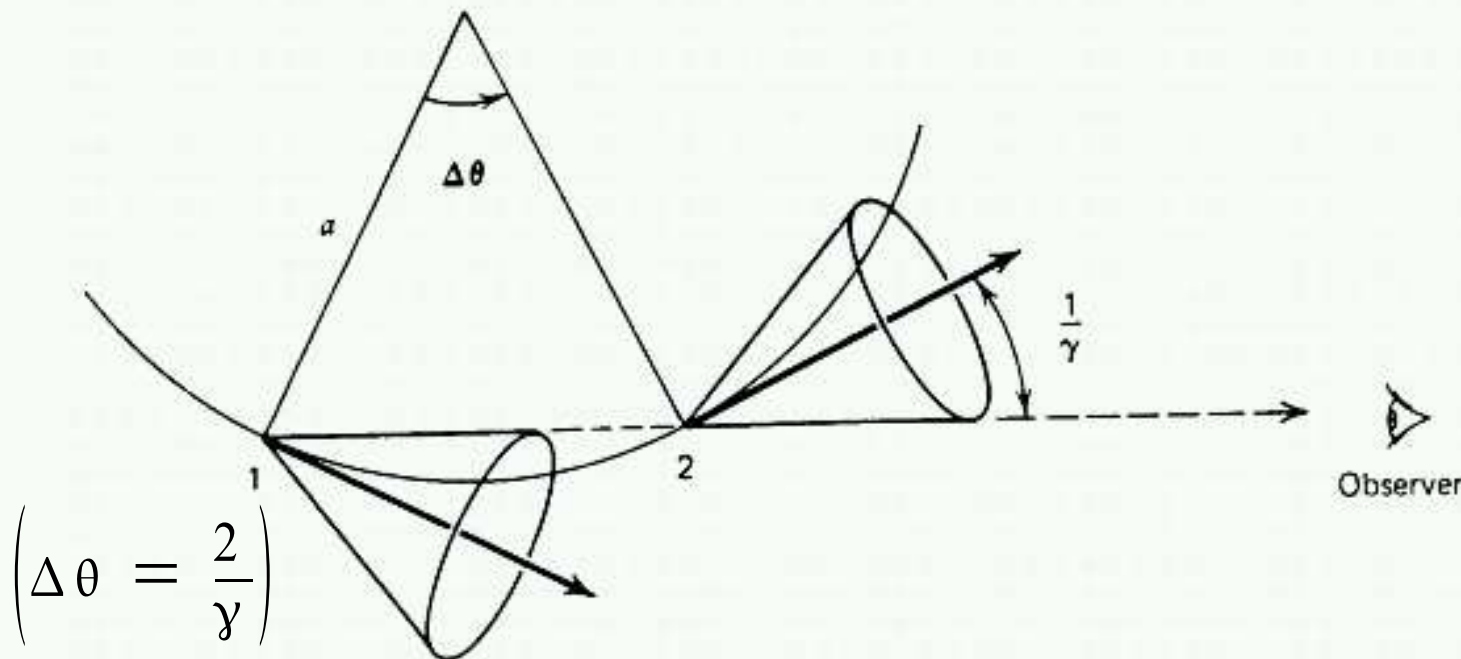
$$\sigma_T = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2 = \frac{8\pi}{3} r_0^2 = 6.6524586 \dots \cdot 10^{-25} \text{ cm}^2$$

Pulse duration in the electron reference frame is

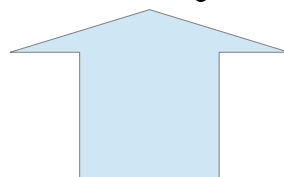
$$\Delta t = \frac{\Delta \theta}{\omega_{\text{rel}}} = \frac{m_e c \gamma}{e H} \frac{2}{\gamma} = \frac{2}{\omega_L}$$

In the observer frame, the duration of the pulse is **shortened by propagation (Doppler) effects**:
the electron is closer to the observer in 2 travelling nearly at same speed of radiation emitted in 1

$$\Delta \tau = \left(1 - \frac{v}{c}\right) \Delta t = (1 - \beta) \Delta t \approx \frac{1}{\gamma^2} \Delta t = \frac{1}{\gamma^2 \omega_L} = \frac{1}{\gamma^3 \omega_{\text{rel}}} \simeq \frac{5 \cdot 10^{-8}}{\gamma^2 H [\text{G}]} \text{ s}$$

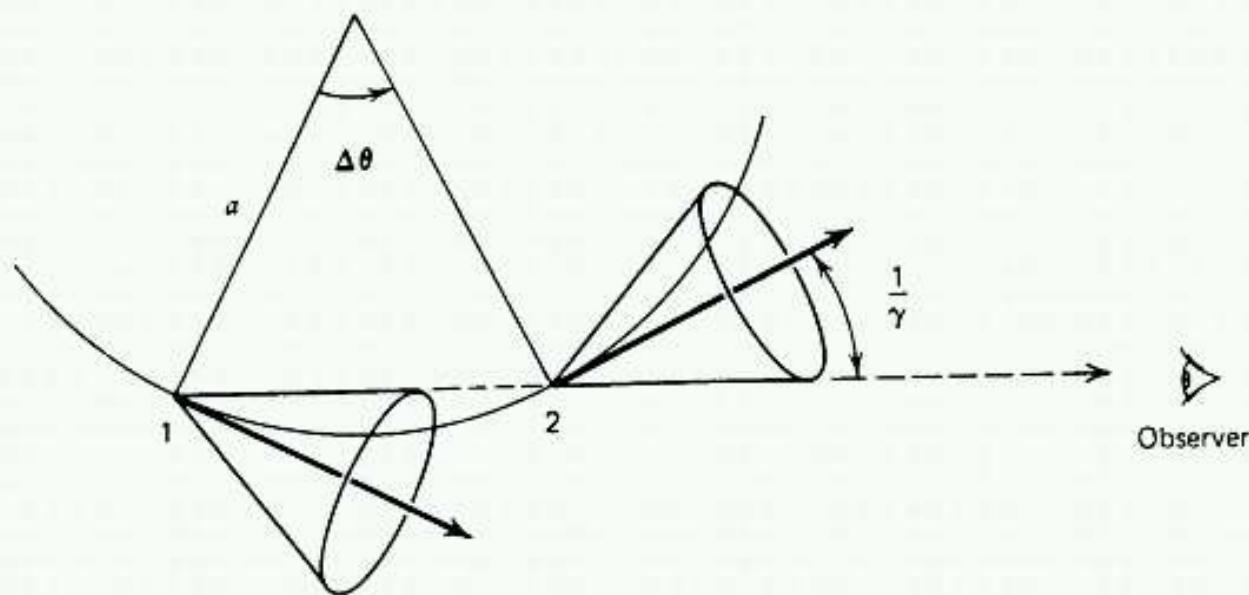
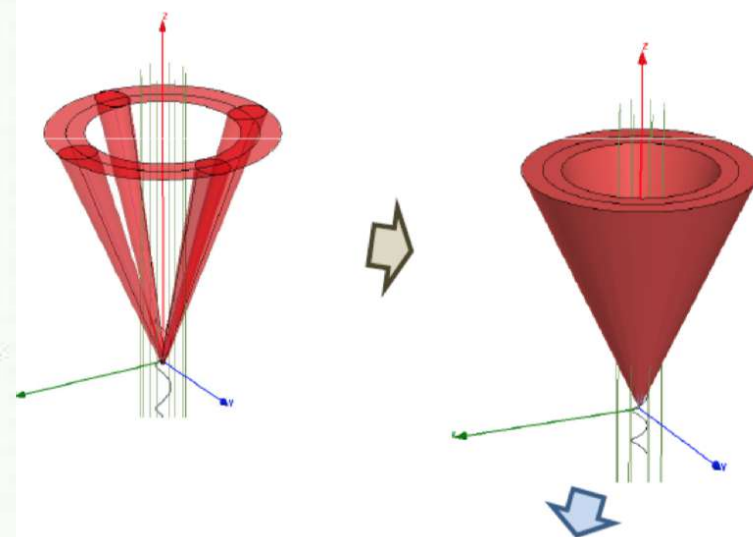


$$\Delta \tau = \left(1 - \frac{v}{c}\right) \Delta t = (1 - \beta) \Delta t \approx \frac{1}{2\gamma^2} \Delta t = \frac{1}{2\gamma^2 \omega_L} = \frac{1}{2\gamma^3 \omega_{\text{rel}}} \approx \frac{5 \cdot 10^{-8}}{\gamma^2 H[\text{G}]} \text{ s}$$

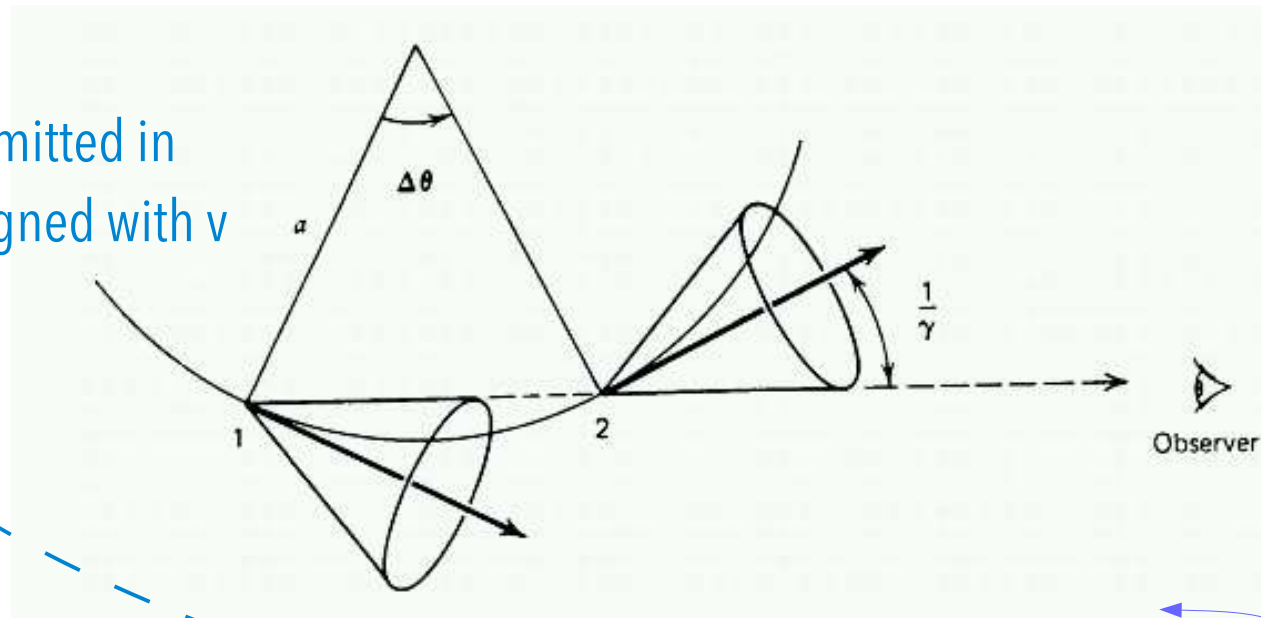


Foreword/consequences:

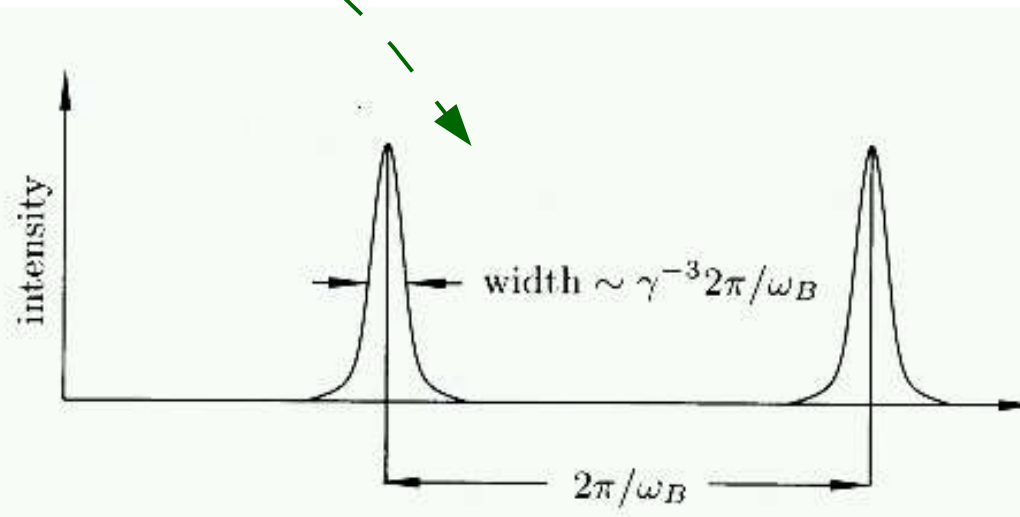
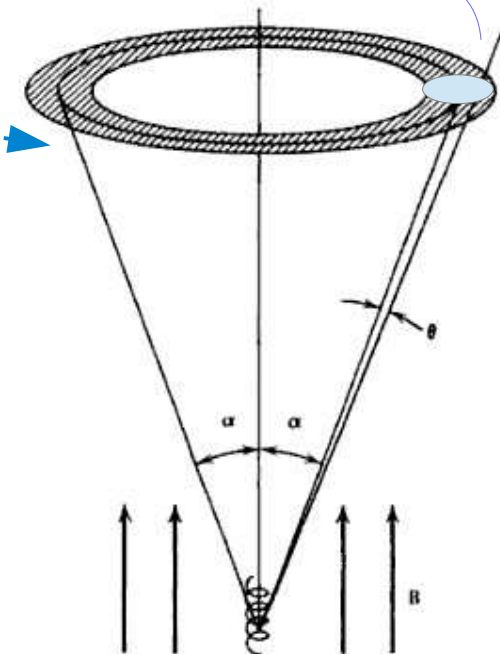
1. The duration of the pulse is $\approx 1/\gamma^2$ shorter than $\Delta t = T_L$
2. The factor $(1 - v/c)$ is the same as appears in the L-W potentials and accounts for the **beaming** (more generally Doppler effect) in case of relative motion wrt the observer (pp. 200-202 Longair)



The radiation is emitted in a narrow cone aligned with $\mathbf{v}$



A given LoS receive pulsed flashes



Instantaneous illumination

Figure 6.5 Synchrotron emission from a particle with pitch angle α . Radiation is confined to the shaded solid angle.



algebra is very complex; a detailed discussion is in Rybicki-Lightman p.175

Fourier analysis of the pulse provides the spectrum of the radiated energy.

The spectrum is the same as the relativistic cyclotron but with an infinite number of harmonics.

Beaming effects move the peak to frequencies higher than ν_L

The characteristic frequency is:

$$\nu_s \simeq \frac{3}{4\pi} \frac{1}{\tau} = \frac{3}{4\pi} \gamma^2 \nu_L = \frac{3}{4\pi} \gamma^2 \frac{eH}{m_e c} =$$
$$\frac{3}{4\pi} \frac{eH}{m_e^3 c^5} \epsilon^2 \simeq 6.24 \cdot 10^{18} \epsilon^2 H \simeq 4.2 \cdot 10^{-9} \gamma^2 H [\mu G] \text{ GHz}$$

⇒ example: an electron with $\gamma \sim 10^4$ in a $H \sim 1 \mu G$ field has $\nu_s \sim 0.4 \text{ GHz}$

Synchrotron radiation is generally emitted in the radio window

H fields are generally 0.1 - 10 μG in space. Higher fields in "condensed" objects (e.g. stars)

The full expression of the emitted energy as a function of frequency is

$$\frac{dE(\nu)}{dt} = \frac{dE_{\parallel}}{dt} + \frac{dE_{\perp}}{dt} \approx \frac{\sqrt{3}e^3 H \sin \theta}{8\pi^2 c m_e} F\left(\frac{\nu}{\nu_s}\right)$$

where

$$F\left(\frac{\nu}{\nu_s}\right) = \frac{\nu}{\nu_s} \int_{\frac{\nu}{\nu_s}}^{\infty} K_{5/3}(y) dy$$

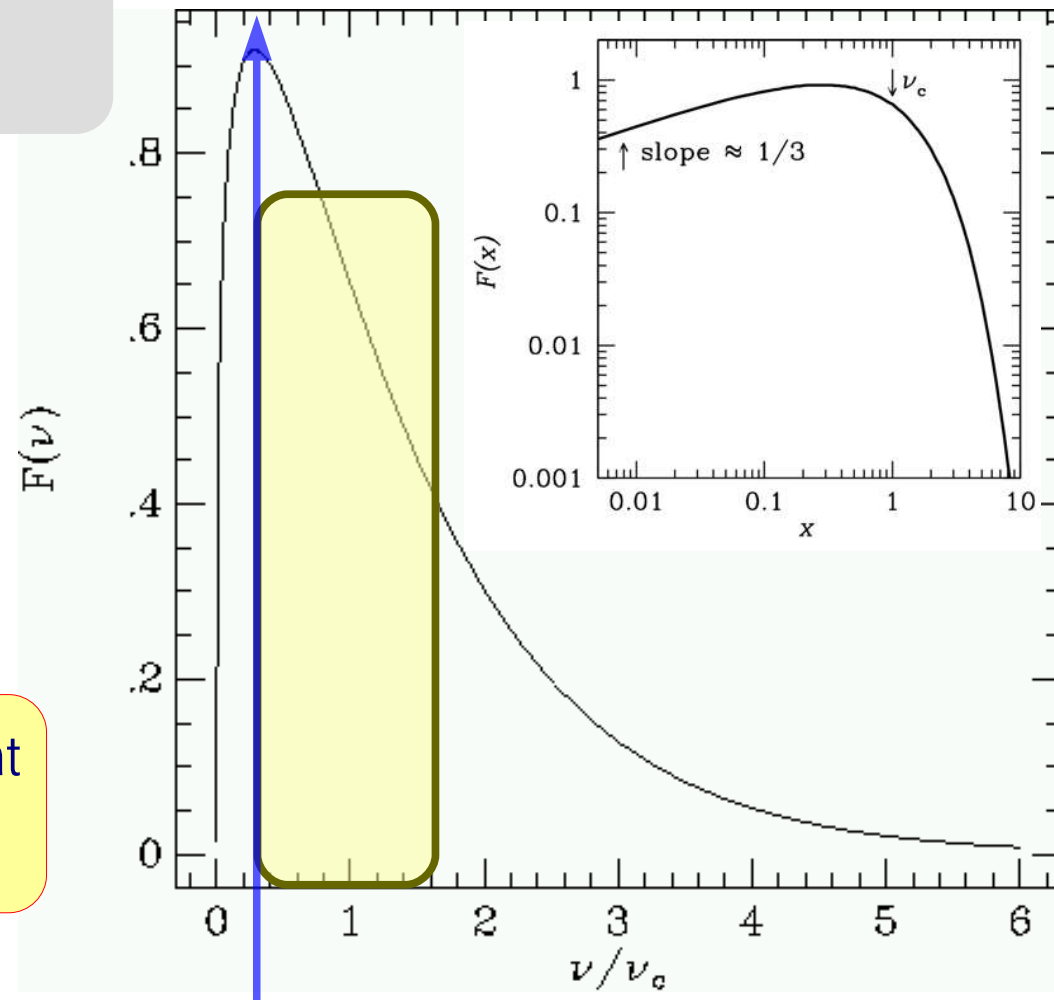
K = modified Bessel function

$$\nu \ll \nu_s \rightarrow F(\nu) \sim \nu^{1/3}$$

$$\nu \gg \nu_s \rightarrow F(\nu) \sim e^{-\nu/\nu_s}$$

peak at $\frac{\nu}{\nu_s} \sim 0.3$

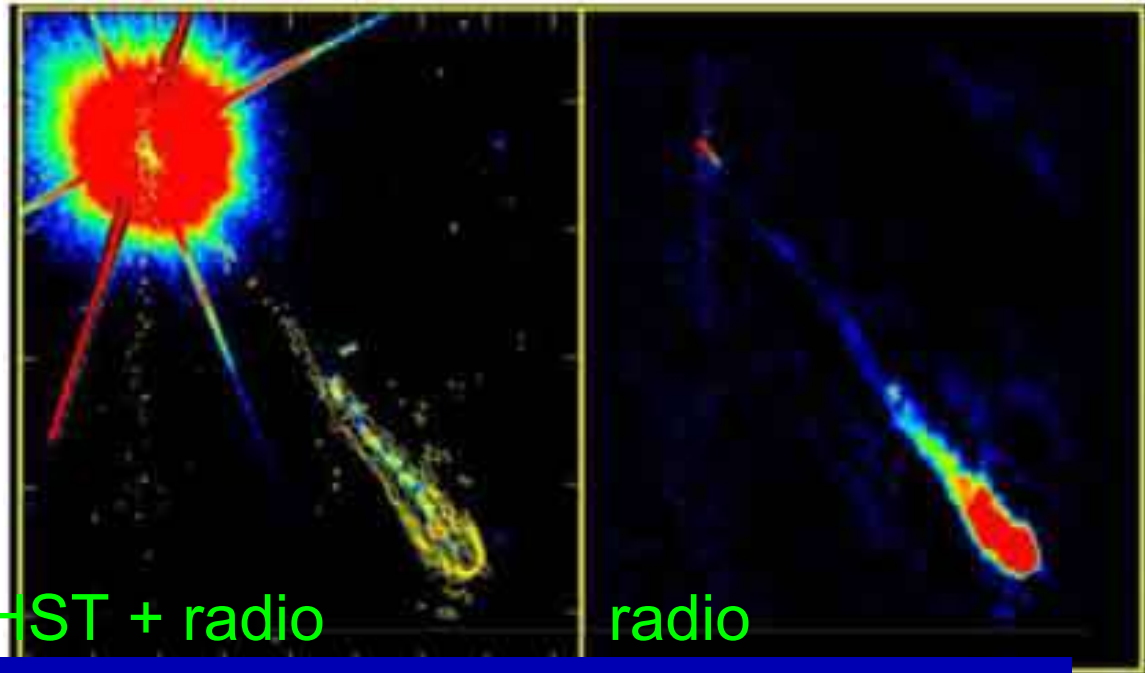
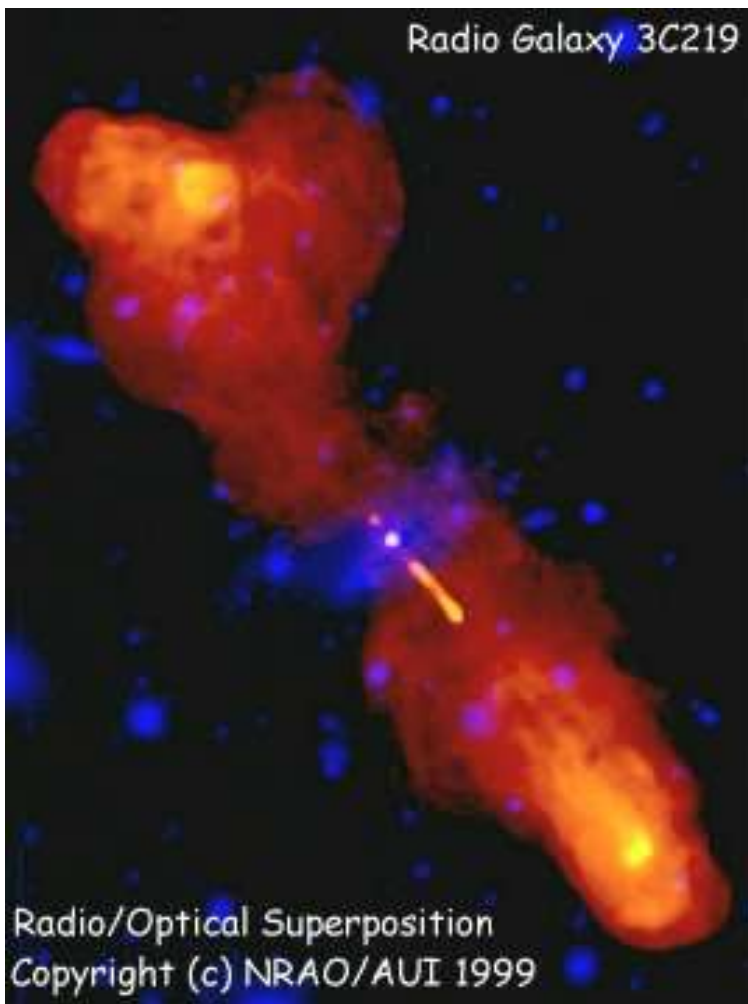
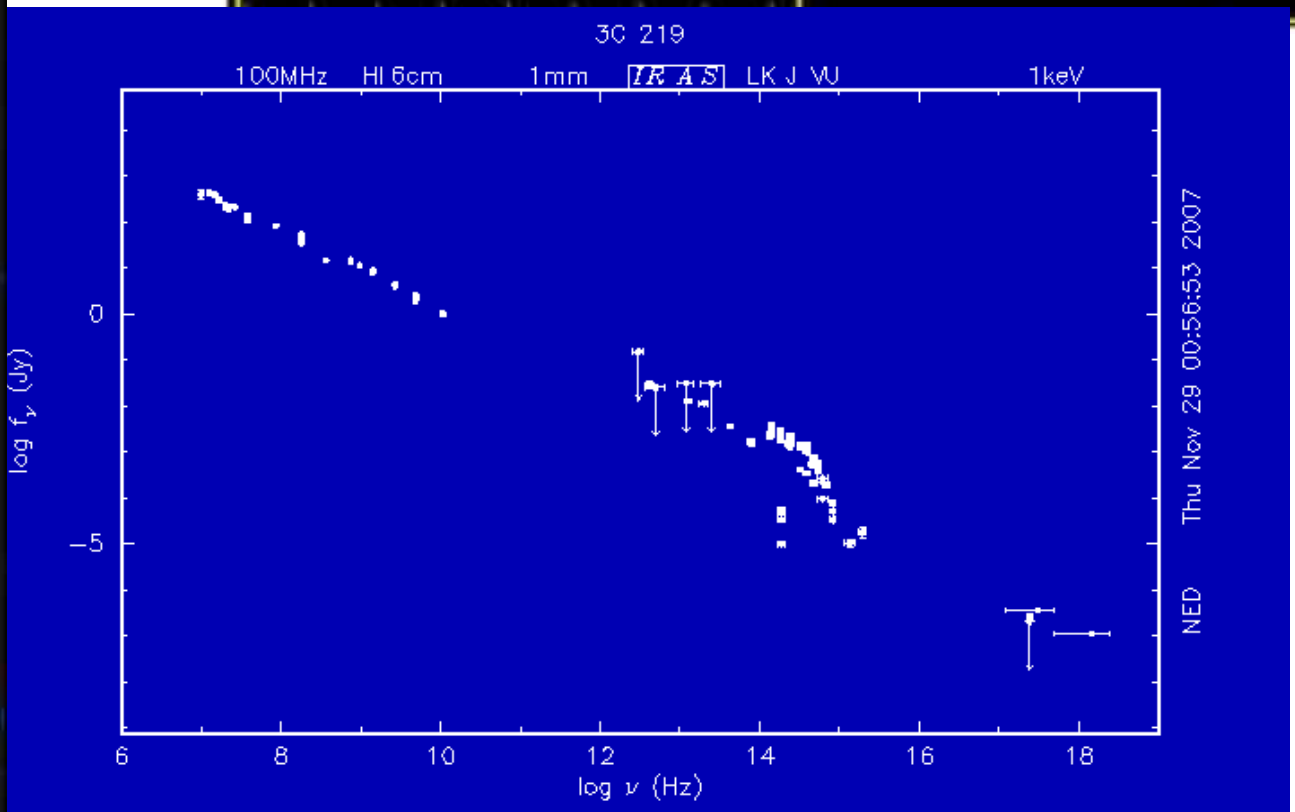
Nearly monochromatic emission, at $\nu_m \sim 0.3 \nu_s$ depending on γ^2 and H



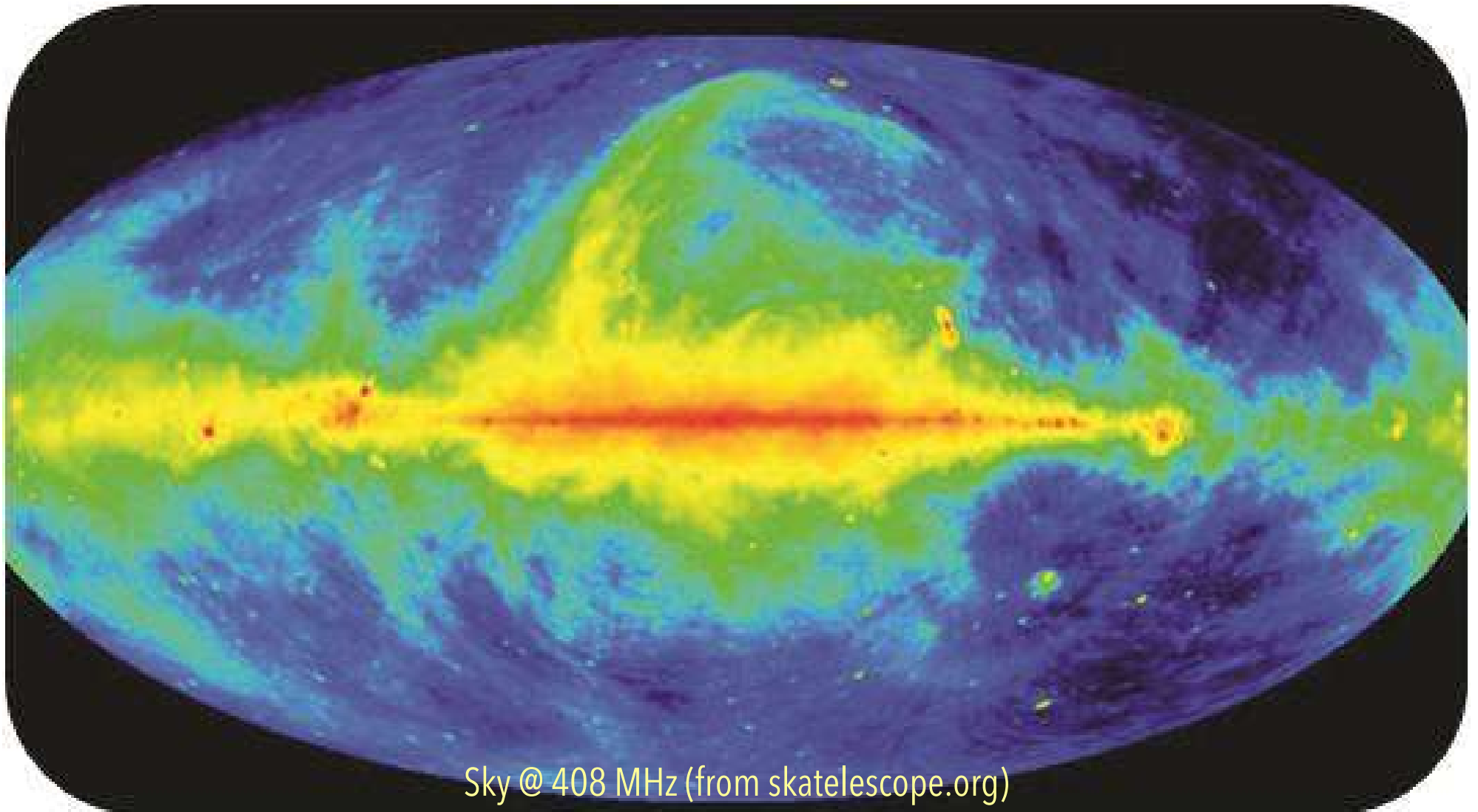


Which is the nature of the emitted energy?

The cases of 3C219 and 3C273



Synchrotron emission from relativistic electrons (positrons) within the Milky Way.
In a field of $1 \mu\text{G}$, their energies correspond to $\gamma \sim 10^4$.



Fast charges within a spiral galaxy: do they escape?

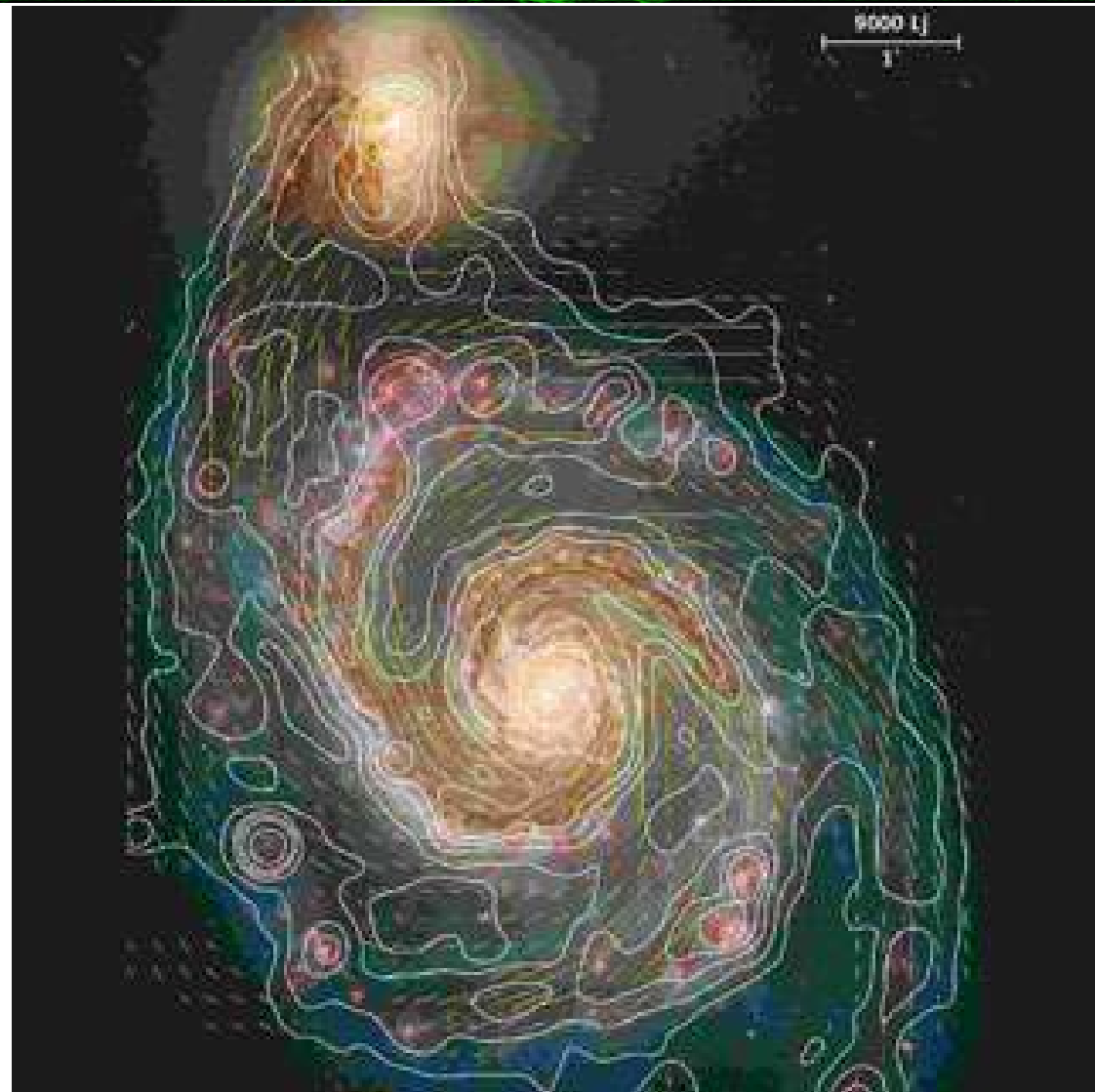
Compute the Larmor radius of a relativistic proton in our galaxy

$$B \sim 10 \mu\text{G}$$

$$r_{\text{L,rel}} = \gamma r_{\text{L}} = \gamma \frac{m v c}{q H}$$

alternative formulas

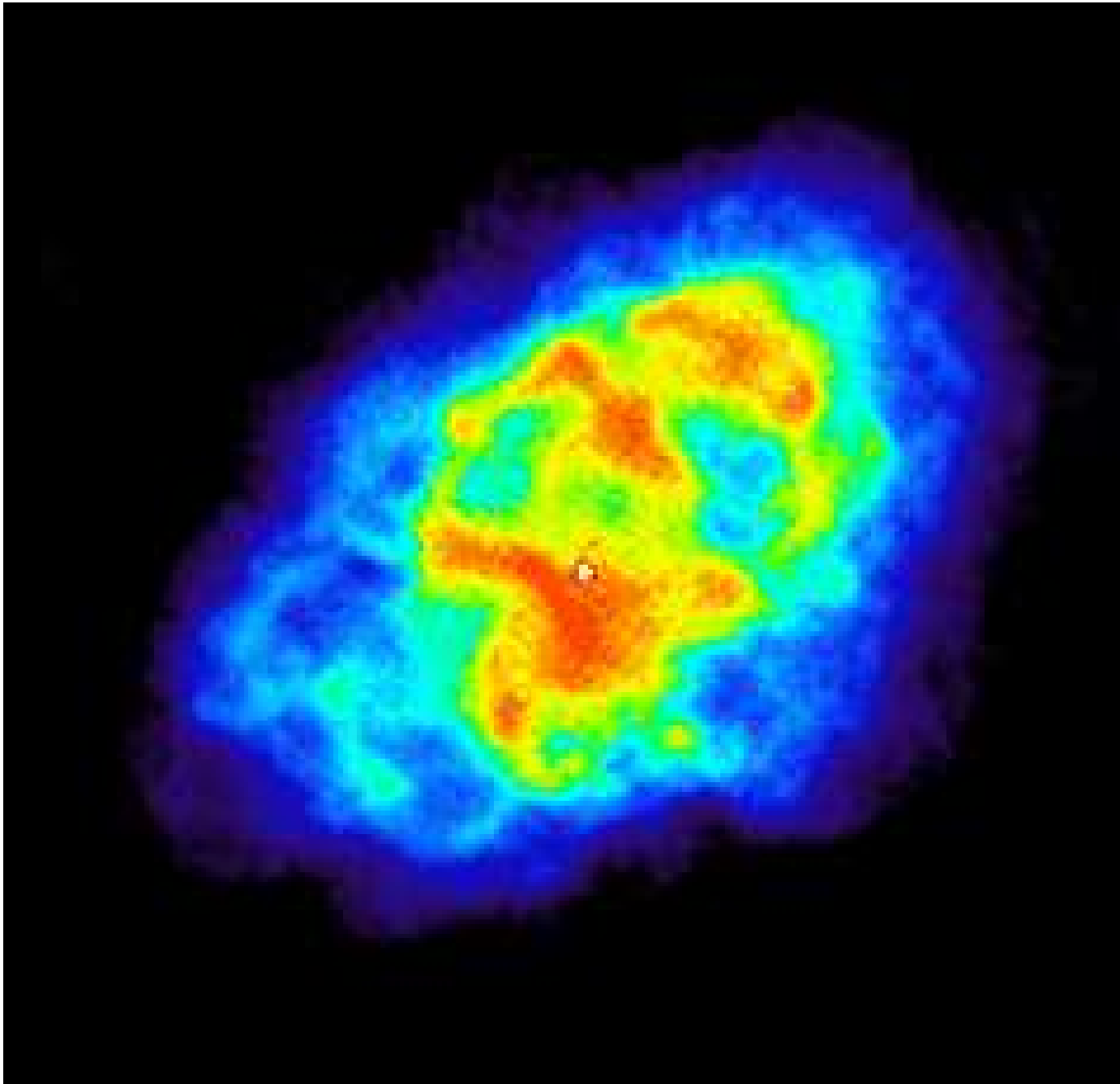
$$r_{\text{L,rel}} = \gamma r_{\text{L}} = \frac{\gamma m c^2 v}{q H c}$$



$$r_{\text{L,rel}} = 3.3 \times 10^6 \frac{\frac{\gamma m c^2}{[\text{GeV}]} \left(\frac{v}{c} \right)}{\frac{q}{[e]} \frac{H}{[\text{Gauss}]}} \text{ cm}$$

In general, relativistic electrons are trapped by magnetic fields in spiral galaxies and emit synchrotron radiation

Synchrotron emission from relativistic electrons (positrons) in the crab nebula (debris of a stellar explosion)



Let's consider an ensemble of relativistic electrons with energies distributed according to a **power – law**:

$$N(\epsilon)d\epsilon = N_0 \epsilon^{-\delta} d\epsilon$$

The specific emissivity of the whole population is

$$J_s(\nu)d(\nu) = \frac{dW_s(\nu, \epsilon)}{dt} N(\epsilon)d\epsilon \quad (****)$$

$$\approx F\left(\frac{\nu}{\nu_s}\right) N_0 \epsilon^{-\delta} d\epsilon$$

There are various (approximate) ways to derive the total emissivity:

1. all the energy is radiated at the characteristic frequency
2. the energy is radiated at a constant rate over a small frequency range

Let's choose approach 1: all the energy is radiated at the characteristic frequency:

$$\nu \simeq \nu_s \approx \gamma^2 \nu_L = \left(\frac{\epsilon}{mc^2} \right)^2 \nu_L \quad \text{where} \quad \nu_L = \frac{eH}{2\pi m_e c} \quad \text{then}$$

$$\epsilon = \gamma m_e c^2 = \left(\frac{\nu}{\nu_L} \right)^{1/2} m_e c^2 = m_e c^2 \nu^{1/2} \nu_L^{-1/2} \quad \text{then, getting the differential}$$

$$d\epsilon = \frac{m_e c^2}{2 \nu_L^{1/2}} \nu^{-1/2} d\nu = \frac{1}{2} m_e c^2 \nu_L^{-1/2} \nu^{-1/2} d\nu$$

Larmor's formula for synchrotron radiation $\left(- \right) \frac{dE}{dt} = \frac{2}{3} \frac{q^4}{m^2 c^3} \beta^2 \gamma^2 H^2 \sin^2 \theta \approx \epsilon^2 H^2 \sin^2 \theta$

Then, for a relativistic (= power-law) population of electrons, (****) becomes.....

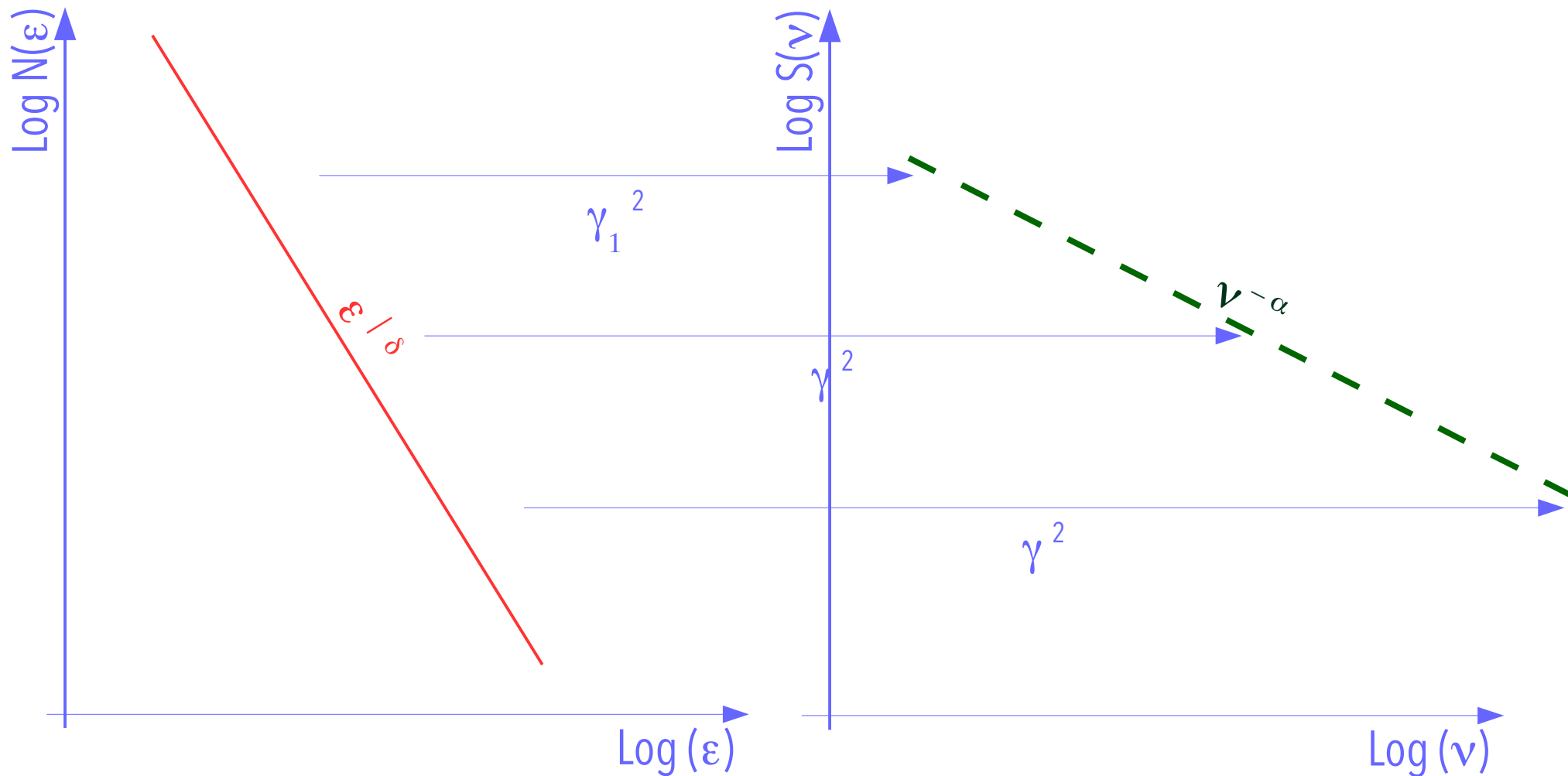
$$J_s(\nu) = \frac{dE_s(\nu, \epsilon)}{dt} N(\epsilon) d\epsilon \approx \text{constant } N_0 H^{(\delta+1)/2} \nu^{-(\delta-1)/2}$$

$$J_s(\nu) \sim N_0 H^{(\delta+1)/2} \nu^{-\alpha}$$

$$\nu_s \simeq 4.2 \cdot 10^{-9} \gamma^2 H [\mu G] \quad \text{GHz}$$

→ Given a magnetic field H ,

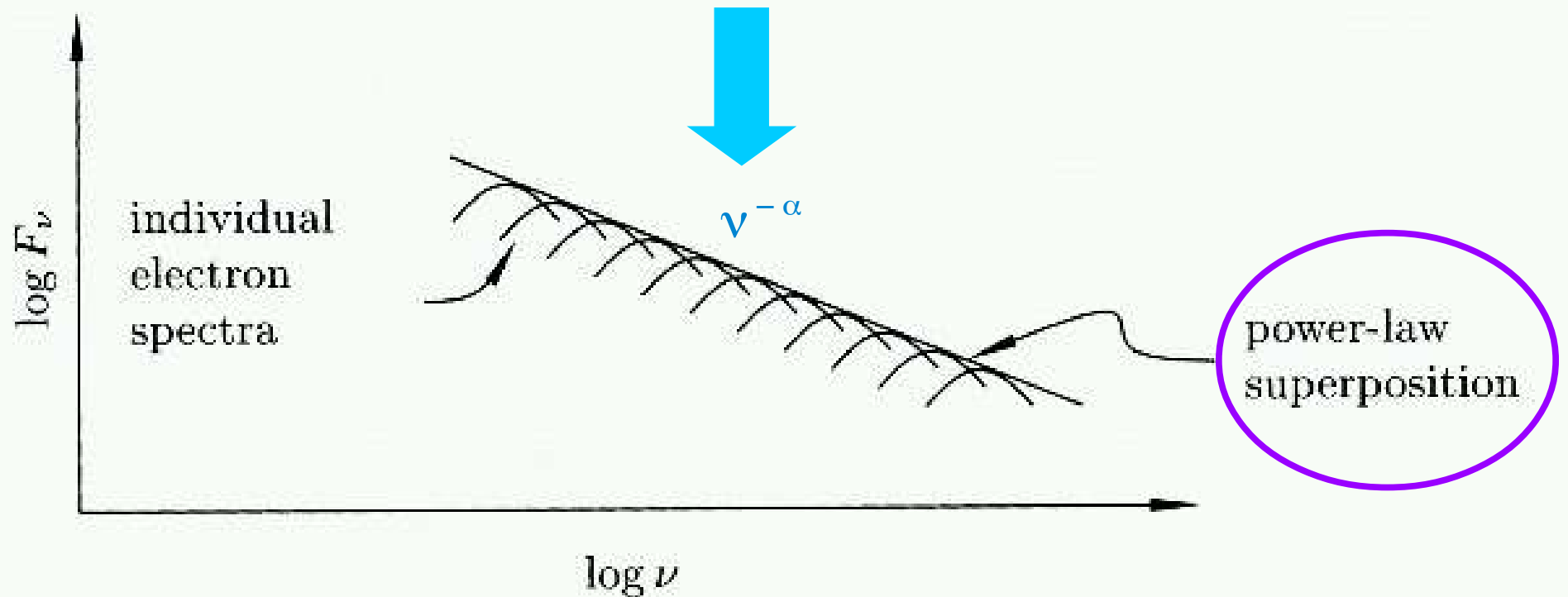
the emitted frequency is determined by γ^2 (i.e. the energy of the charge, being $\varepsilon = \gamma m_e c^2$)



The total spectrum is interpreted as the superposition of many contributions from the various electrons emitting each at its own characteristic frequency ν_s in the same field H

$$\alpha = \frac{\delta - 1}{2}$$

spectral index of the synchrotron radiation





Synchrotron self-absorption (1): (SSA)

"(Low energy) Photons cannot escape regions with high electron densities"

NOT in thermal equilibrium, then **Kirchoff's law does not apply.**

Concept of temperature must be widened, it does not imply thermal equilibrium

The "electron (kinetic) temperature" T_e can be defined based on the electron energy

SSA applies when $T_B \sim T_e$

T_B can NEVER EXCEED T_e

FYI,
Unnecessary
detail here

The absorption coefficient is effectively determined by making use of the Einstein's coefficients (not given here, see further lectures)

$$\mu_s \approx N_0 \nu^{-(\delta+4)/2} H_{\perp}^{(\delta+2)/2}$$

see Rybicky-Lightman, 6.8



Synchrotron self-absorption (2)

⇒ Synchrotron absorption

$$\mu_s \approx N_0 \nu^{-(\delta+4)/2} H_{\perp}^{(\delta+2)/2}$$

⇒ Synchrotron emissivity

$$J_s(\nu) \approx \nu^{-\frac{\delta-1}{2}} H_{\perp}^{\frac{\delta+1}{2}}$$

To be inserted in the radiative transfer equation of an emitting cloud (partially) opaque

$$B_s = \frac{J_s(\nu)}{4\pi\mu_s(\nu)} (1 - e^{-\tau_s(\nu)})$$

Which becomes

$$B_s(\nu) \approx \nu^{-\frac{\delta-1}{2}} H_{\perp}^{\frac{\delta+1}{2}} \quad \tau \ll 1 \text{ optically thin regime}$$

$$B_s(\nu) \approx \nu^{5/2} H_{\perp}^{-1/2} \quad \tau \gg 1 \text{ optically thick regime}$$

N.B. The first case is that derived from the earlier description which provides a power-law distribution



Use a trick: the power-law distribution can be seen as the superposition of many Maxwellian distributions at different T : $\gamma m_e c^2 = 3kT$ (for a relativistic plasma)

Let's suppose that each electron emits at its own characteristic frequency ν_s

$$\nu_s \simeq 4.2 \cdot 10^{-9} \gamma^2 H [\mu G] \text{ GHz}$$

FYI,
Skipping explanation

From this we get $kT \sim \gamma m_e c^2 \sim m_e c^2 \left(\frac{\nu}{\nu_L} \right)^{1/2} H^{-1/2} \rightarrow T \sim \nu^{1/2}$

At low energies, in a fully opaque source (black body!) the brightness temperature is defined as

$$I(\nu) = B(\nu, T) \stackrel{\text{def}}{=} 2kT_B \frac{\nu^2}{c^2} \quad T_B \text{ must be = to the kinetic temperature of electrons}$$

$$I(\nu) \sim m_e c^2 \left(\frac{\nu}{\nu_s} \right)^{1/2} H^{-1/2} \frac{\nu^2}{c^2} \sim \frac{\nu^{5/2}}{H^{1/2}}$$

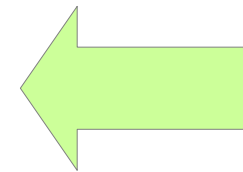
The brightness temperature of a region does NEVER exceed the electron temperature:

$$T_B \simeq \frac{\lambda^2}{2k} \frac{S(\nu)}{\theta_1 \cdot \theta_2} \quad \text{but} \quad S(\nu) \approx \nu^{-\alpha} \quad \text{and} \quad 3kT_e = \gamma m_e c^2$$

$$T_{B, \max} = 3.0 \cdot 10^{10} \cdot \frac{\nu_{\max}^{1/2}}{[\text{GHz}]} \cdot \frac{H^{-1/2}}{[\text{G}]} \quad [\text{K}]$$

Observations may find T_B in excess of this maximum value

→ Further considerations are necessary (later on...)

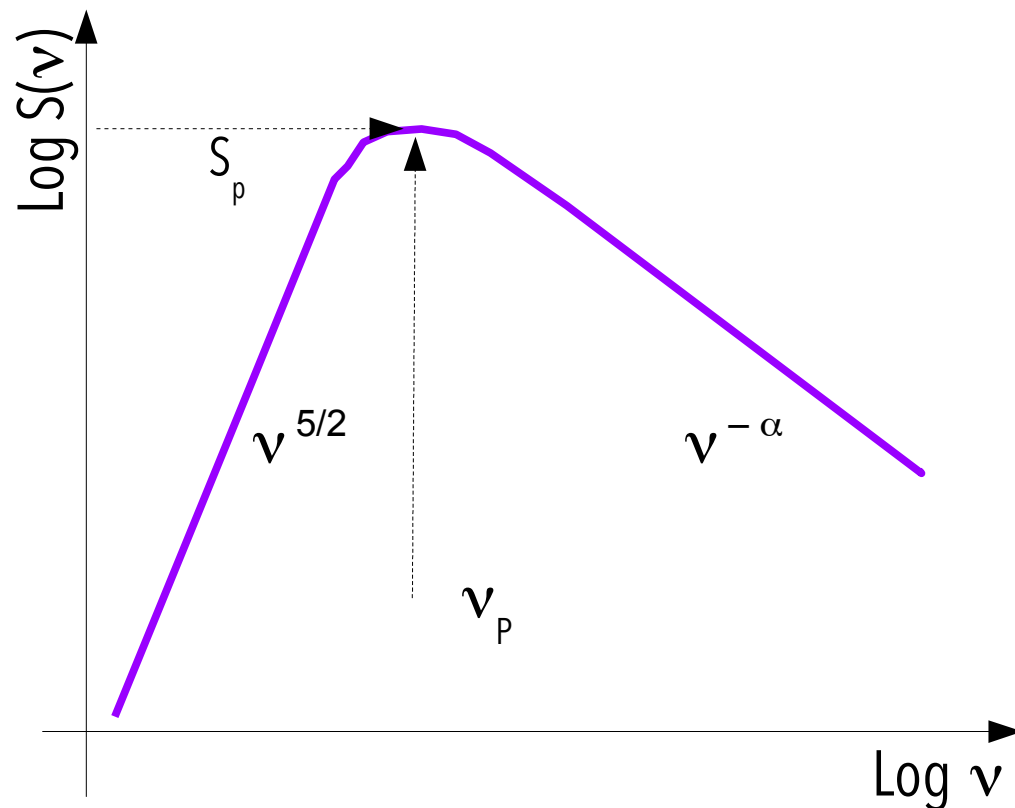
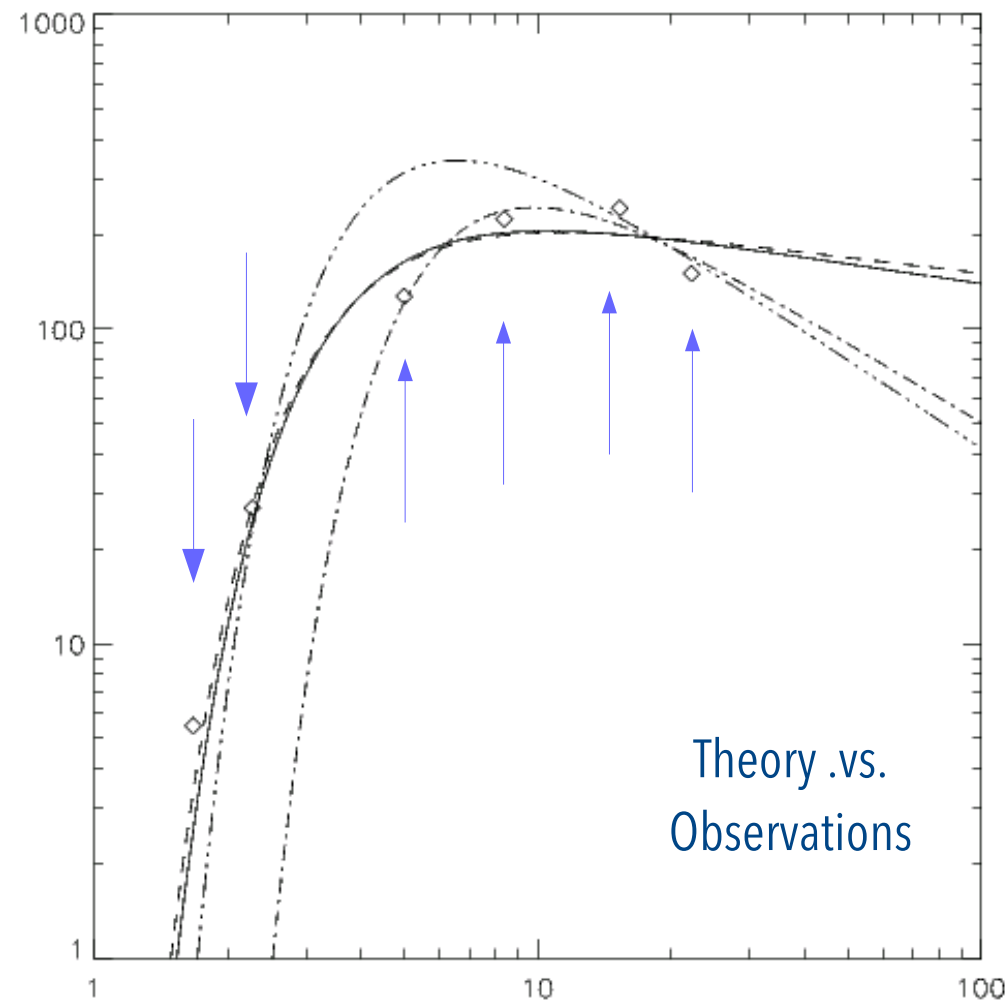


Synchrotron self-absorption (5):

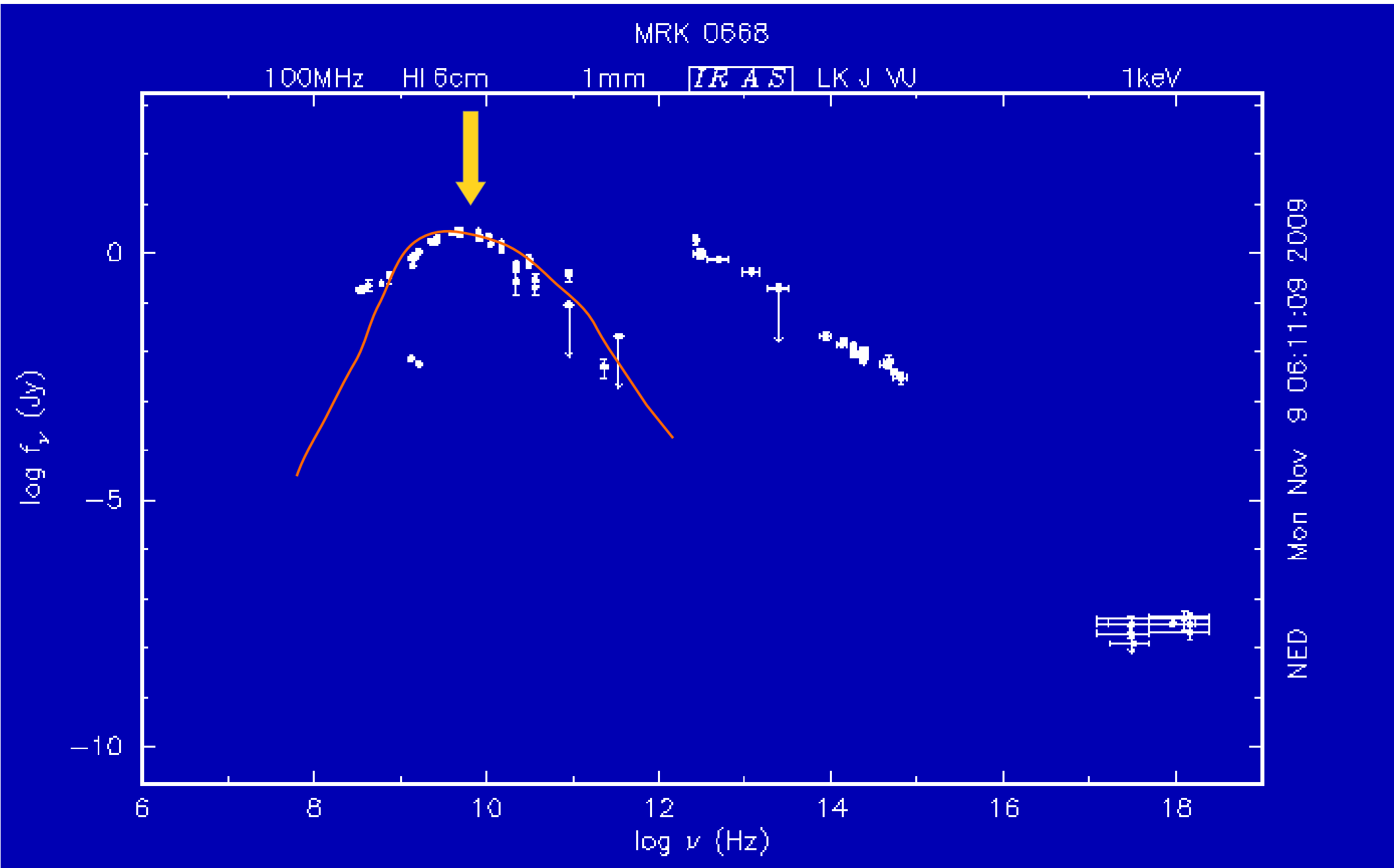
Let's derive the expression for J_s to get the peak values:

$$\left(\frac{\nu_p}{\text{GHz}} \right) \simeq \text{const} \left(\frac{S_p}{\text{Jy}} \right)^{2/5} \left(\frac{\theta}{\text{mas}} \right)^{-4/5} \left(\frac{H_{\perp}}{\text{mG}} \right)^{1/5} (1+z)^{1/5}$$

With these units, the constant is ≈ 2

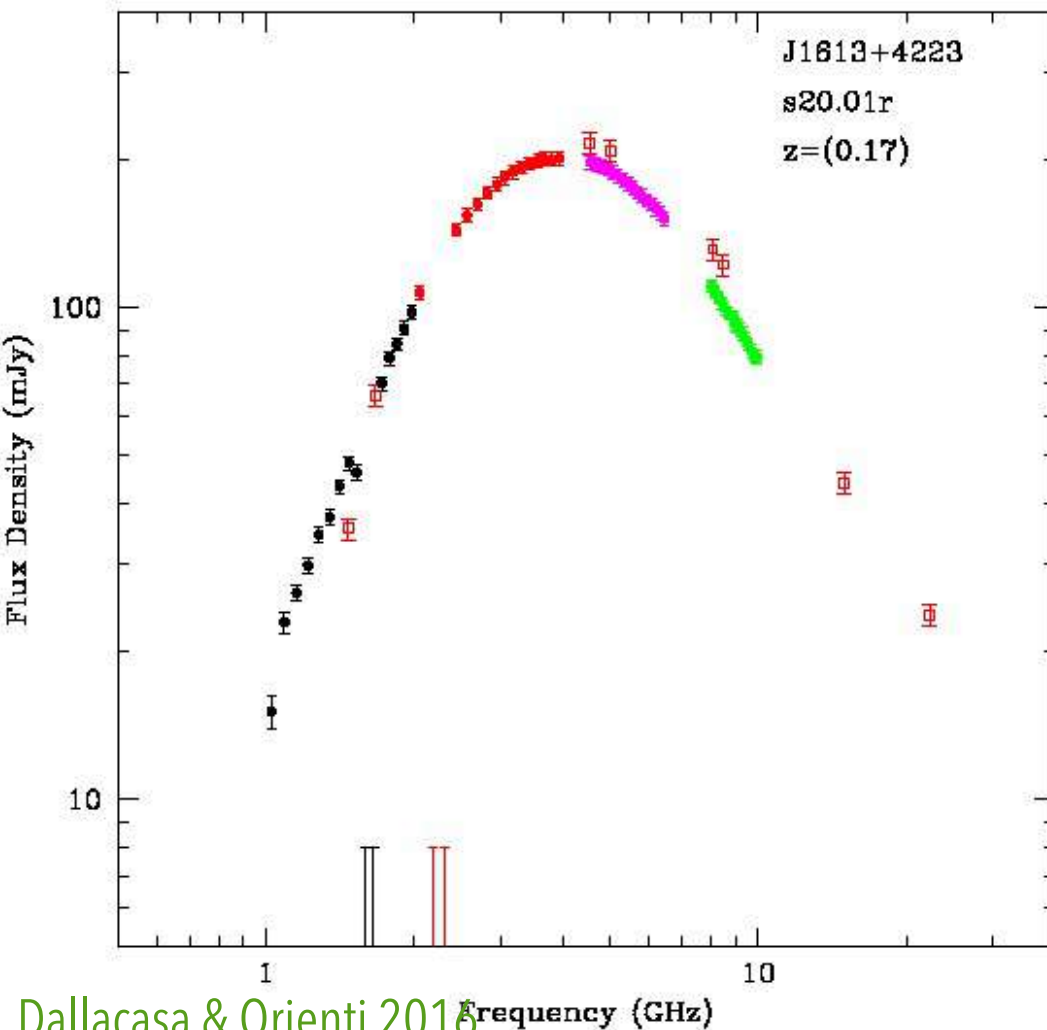
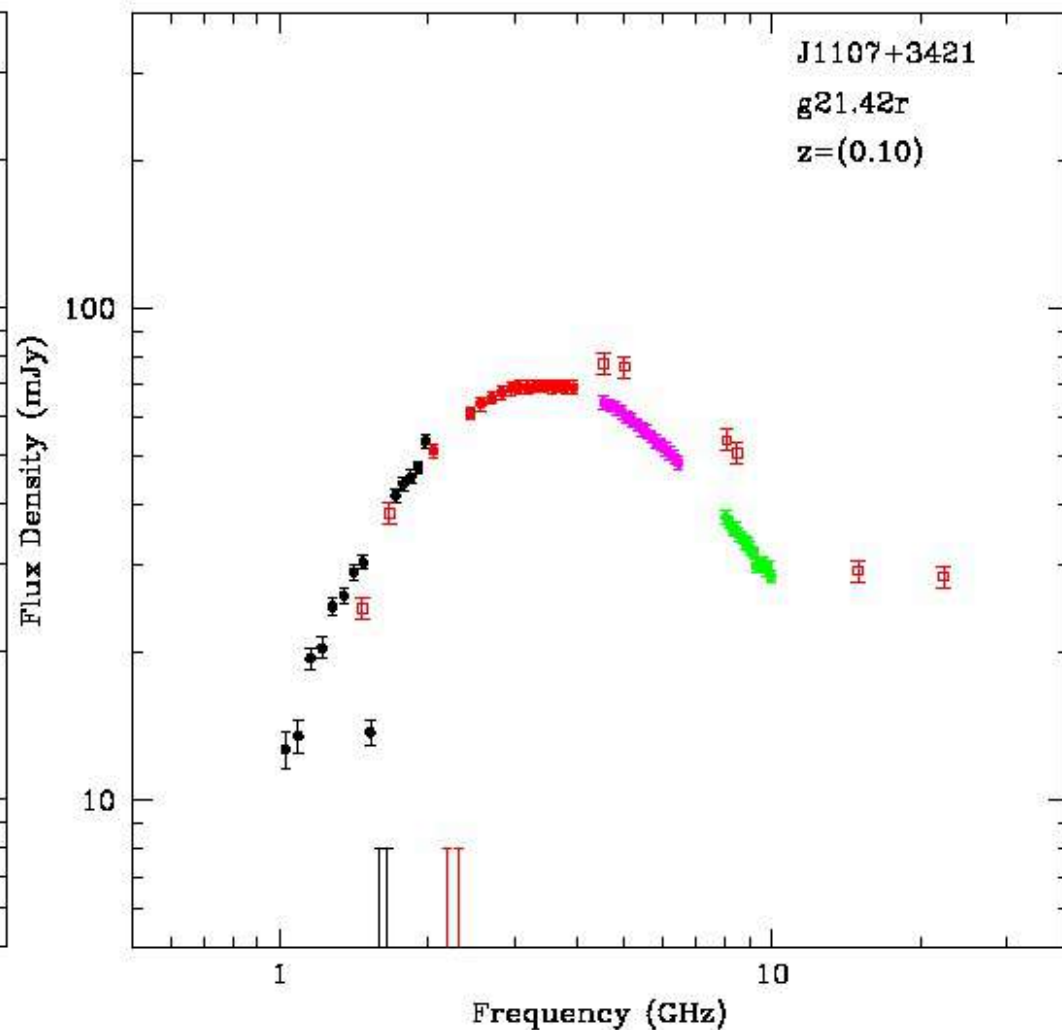


Synchrotron self-absorption (6): A peaked radio spectrum



$$\nu_p \approx S_p^{2/5} \theta^{-4/5} H_{\perp}^{1/5} (1+z)^{1/5}$$

Can help in measuring the magnetic field in the relativistic plasma!!!!

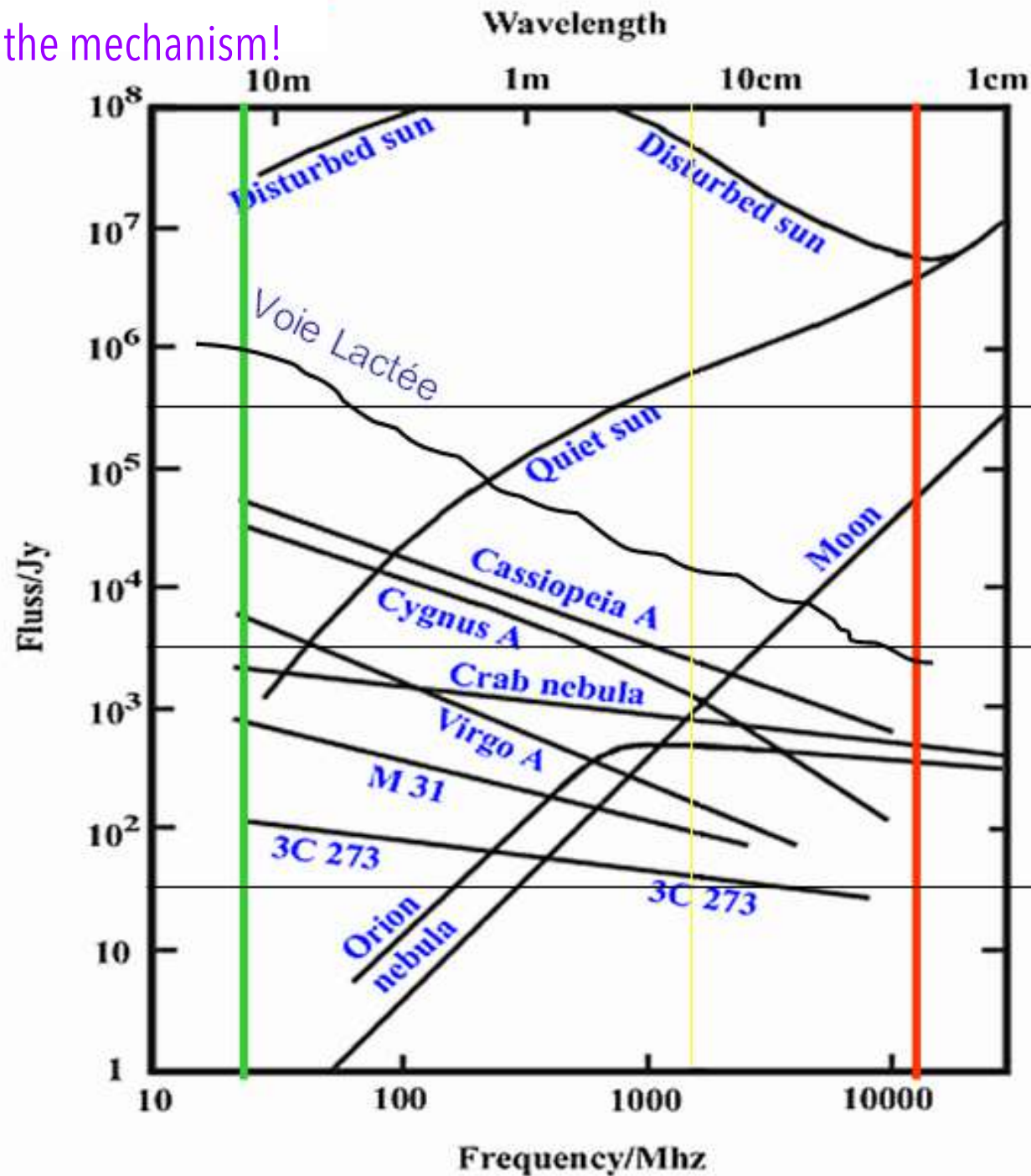




Spectra of radio sources:

(none of these is "compact"!)

Test: Guess the mechanism!





Energetics of a radio source (1)

Total energy of a synchrotron emitting body must take into account both particles and the H field

$$U_{\text{tot}} = \epsilon_{\text{el}} + \epsilon_{\text{pr}} + U_{\text{H}} = (1+k)\epsilon_{\text{el}} + U_{\text{H}} = U_{\text{p}} + U_{\text{H}}$$

k is a measure of our "ignorance" of the plasma composition

Let's consider relativistic electrons: their energy is given by

$$\epsilon_{\text{el}} = V \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} \epsilon N(\epsilon) d\epsilon = V N_0 \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} \epsilon \epsilon^{-\delta} d\epsilon = \frac{N_0 V}{2-\delta} \left(\epsilon_{\text{max}}^{2-\delta} - \epsilon_{\text{min}}^{2-\delta} \right) \quad \delta \neq 2$$

We can get rid of $N_0 V$ if we consider the source Lumonosity

$$L = 4\pi D^2 \int_{\nu_{\text{min}}}^{\nu_{\text{max}}} S(\nu) d\nu = V \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} N(\epsilon) \frac{d\epsilon}{dt} d\epsilon$$

$$L \approx N_0 V \int_{\epsilon_{\text{min}}}^{\epsilon_{\text{max}}} \epsilon^{-\delta} H^2 \epsilon^2 \sin^2 \theta d\epsilon = 2.4 \cdot 10^{-3} \frac{N_0 V H^2 \sin^2 \theta}{3-\delta} \left(\epsilon_{\text{max}}^{3-\delta} - \epsilon_{\text{min}}^{3-\delta} \right)$$

$$\text{where } \epsilon_{\text{min/max}} = \left(\frac{\nu_{\text{min/max}}}{6.24 \cdot 10^{18} \text{ H}} \right)^{1/2}$$

Now let's get $N_0 V$ out of this relation



Energetics of a radio source (2)

$$L = 2.4 \cdot 10^{-3} \frac{N_0 V H^2 \sin^2 \theta}{3 - \delta} \left(\epsilon_{\max}^{3-\delta} - \epsilon_{\min}^{3-\delta} \right) \Rightarrow N_0 V = \frac{(3 - \delta) L}{2.4 \cdot 10^{-3} H^2 \sin^2 \theta \left(\epsilon_{\max}^{3-\delta} - \epsilon_{\min}^{3-\delta} \right)}$$

$$\begin{aligned} \text{If } \epsilon_{\text{el}} &= \frac{3 - \delta}{2 - \delta} \frac{\epsilon_{\max}^{2-\delta} - \epsilon_{\min}^{2-\delta}}{\epsilon_{\max}^{3-\delta} - \epsilon_{\min}^{3-\delta}} \frac{L}{H^2 \sin^2 \theta} \frac{1}{2.4 \cdot 10^{-3}} \approx \epsilon \simeq 4.0 \cdot 10^{-10} H^{-1/2} \nu^{-1/2} \\ &\approx \frac{3 - \delta}{2 - \delta} \frac{[\nu_{\max}^{(2-\delta)/2} - \nu_{\min}^{(2-\delta)/2}] H^{-(2-\delta)/2} \sin^{-(2-\delta)} \theta}{[\nu_{\max}^{(3-\delta)/2} - \nu_{\min}^{(3-\delta)/2}] H^{-(3-\delta)/2} \sin^{-(3-\delta)} \theta} \frac{L}{H^2 \sin^2 \theta} \end{aligned}$$

Once chosen the minimum & maximum energy, considering an isotropic pitch angle distribution

$$\rightarrow H^2 \sin^2 \theta = \frac{2}{3} H^2$$

The above becomes

$$\epsilon_{\text{el}} = C_{\text{el}} H^{-3/2} L$$

It is the energy associated with the relativistic particles (electrons) radiating synchrotron emission

Energy is stored in the magnetic field as well

$$U_H = \int \frac{H^2}{8\pi} dV = C_H H^2 V$$

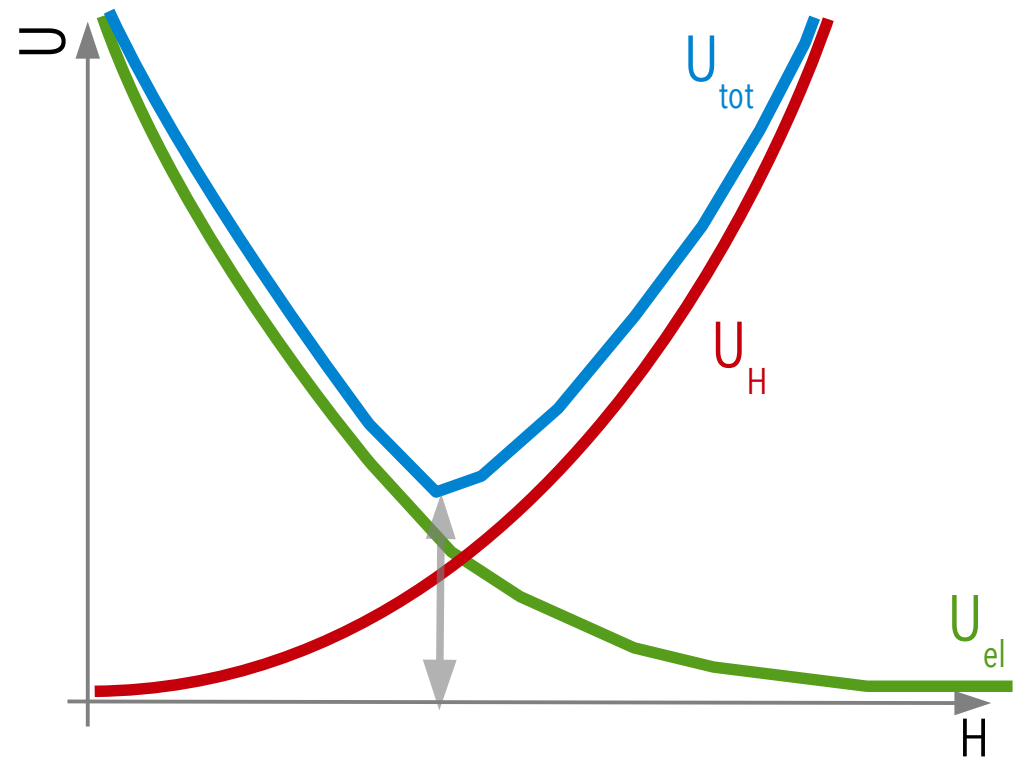
The total energy budget is

$$U_{\text{tot}} = (1 + k) C_{\text{el}} H^{-3/2} L + C_H H^2 V$$

By deriving wrt H

→ a minimum in the total energy content of a synchrotron emitting region!

$$(1 + k) \epsilon_{\text{el}} = \frac{4}{3} U_H$$





"Equipartition" Magnetic Field

$$H_{\text{eq}} = H(\epsilon)_{\text{min}} = \left[\frac{3}{4} (1+k) \frac{C_{\text{el}}}{C_{\text{H}}} \right]^{2/7} \left(\frac{L}{V} \right)^{2/7}$$

Such **equipartition field** provides the value of the field intensity for which the total energy content in the radio source is minimum (related to the relativistic plasma), which amounts to

$$U_{\text{tot, min}} = \frac{7}{4} (1+k) \epsilon_{\text{el}} = \frac{7}{4} (1+k) C_{\text{el}} H_{\text{eq}}^{-3/2} L = 2 [(1+k) C_{\text{el}}]^{4/7} C_{\text{H}}^{3/7} L^{4/7} V^{3/7}$$

It can be calculated from observations, although a number of assumptions are hidden

1. The particle density is homogeneous (filling factor $\phi = 1$)
2. The magnetic field is uniform all over the relativistic plasma

The total energy can be written in a specific expression for radio emission at 1.4 GHz as

$$U_{\text{tot, min}} = 2 \cdot 10^{41} (1+k)^{4/7} \left(\frac{L_{1.4\text{GHz}}}{[\text{Watt}]} \right)^{4/7} \left(\frac{V}{[\text{kpc}]^3} \right)^{3/7} \quad [\text{erg}]$$

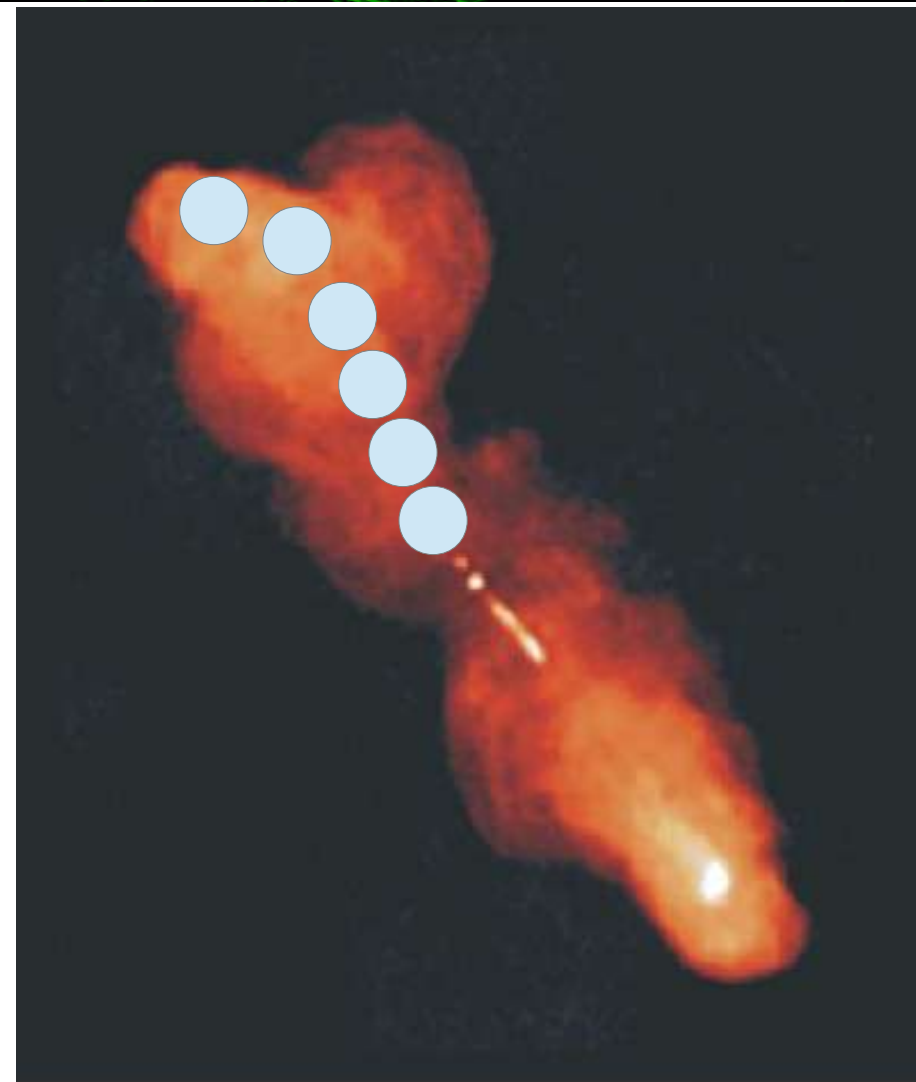
This can be used to define the **energy density** and the **internal pressure** as

$$u_{\text{min}} = \frac{U_{\text{tot, min}}}{V} = 6.8 \cdot 10^{-24} \left(\frac{L_{1.4\text{GHz}}}{[\text{Watt}]} \frac{[\text{kpc}]^3}{V} \right)^{4/7} \quad [\text{erg cm}^{-3}]$$

$$P_{\text{min}} = P_{\text{H}} + P_{\text{rel}} = \frac{H^2}{8\pi} + (\Gamma - 1) u_{\text{rel}} = \frac{11}{21} u_{\text{min}}$$

E.g. $\Rightarrow$ Such pressure can be compared to that of the surrounding ambient medium

- Physical quantities can be calculated either **globally** for the whole source, or **locally**, at various sites within the same radio source:
- Determination of the "**equipartition field**" along the radio source structure (plasma evolution?)
- **Pressure balance**: P_{int} and $P_{\text{ISM}} / P_{\text{ICM}}$ provide the ram pressure and then the expansion/growth velocity (supersonic growth?)



Observers: Alan Biddle, et al.

$$H_{\text{eq}} = H(\varepsilon)_{\text{min}} = \left[\frac{3}{4} (1+k) \frac{C_{\text{el}}}{C_{\text{H}}} \right]^{2/7} \left(\frac{L}{V} \right)^{2/7}$$

$$u_{\text{min}} = \frac{U_{\text{tot,min}}}{V} = 6.8 \cdot 10^{-24} \left(\frac{L_{1.4\text{GHz}} [\text{kpc}]^3}{[\text{Watt}] V} \right)^{4/7} \quad [\text{erg cm}^{-3}]$$

$$P_{\text{min}} = P_{\text{H}} + P_{\text{rel}} = \frac{H^2}{8\pi} + (\Gamma - 1) u_{\text{rel}} = \frac{11}{21} u_{\text{min}}$$

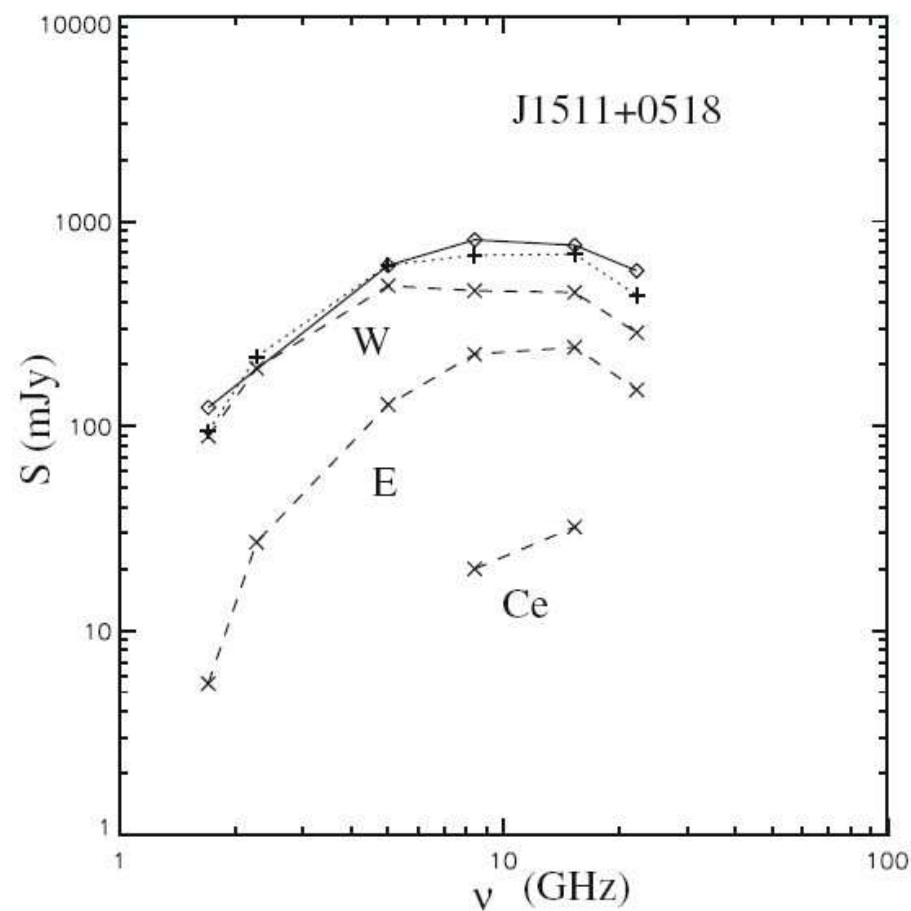
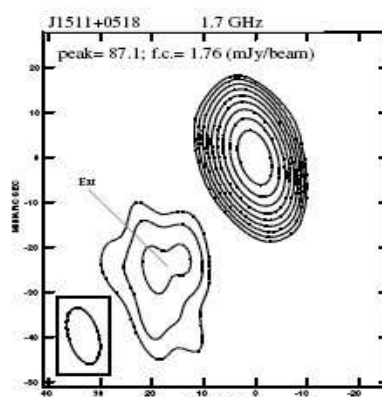
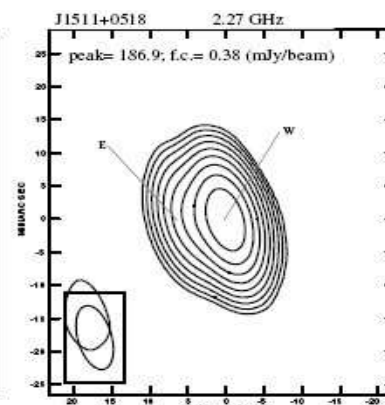
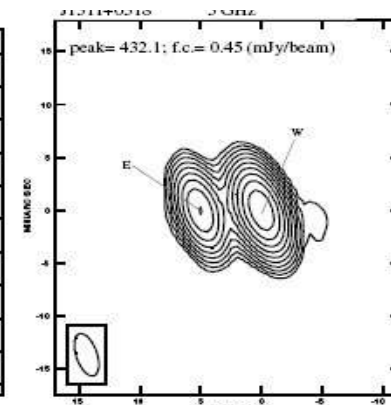
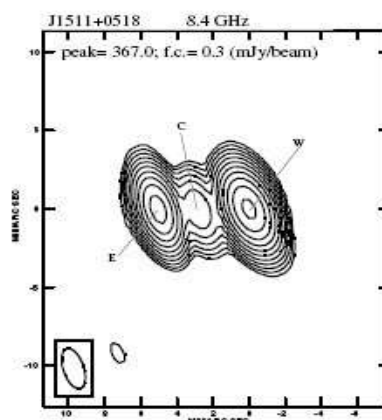
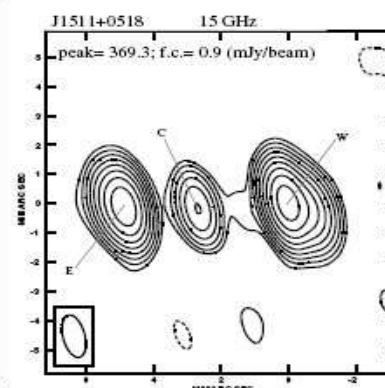
Is equipartition Magnetic Field representative?

H_{eq} measured from Luminosity, Volume

Peaked Spectrum (Self-absorption)
measured from ν_p, S_p, θ (angular size)

Table 2. Physical parameters of the source components.

Source	C	H mG	H_{eq} mG	u_{min} erg/cm ³ 10 ⁻⁴	p_{min} dyne/cm ² 10 ⁻⁴
J0003+2129	E	33	30	5.0	3.1
J0005+0524	E	—	18	0.75	0.46
J0428+3259	E	$\gg 1000$	34	0.75	0.46
J0650+6001	Ce	59	65	3.9	2.4
	N	29	77	6.0	4.0
J1511+0518	S	10	54	1.5	1.0
	W	104	95	8.3	5.2
J1459+3337	E	$\gg 1000$	70	3.8	2.4
		160	160	24	15





Particles radiate at expenses of their kinetic energy. Hence, the energy distribution of a synchrotron emitting region within a given volume V fully filled by magnetized relativistic plasma will change with time

$$\begin{array}{ccccccc} \text{Particle Flow} & & \text{Energy Losses} & & \text{Leakage} & & \text{Injection} \\ \frac{\partial N(\epsilon, t)}{\partial t} & + & \frac{\partial}{\partial \epsilon} \left(\frac{d\epsilon}{dt} N(\epsilon, t) \right) & + & \frac{N(\epsilon, t)}{T_{\text{conf}}} & = & Q(\epsilon, t) \end{array}$$

In particular

$$N(\epsilon, 0) = N_0 \epsilon^{-\delta}$$

$$Q(\epsilon, t) = A \epsilon^{-\delta}$$

First considerations in a simple case: $T_{\text{conf}} = \infty$ & $Q(\epsilon, t) = 0$



(pure) Radiative losses:

$$T_{\text{conf}} = \infty \quad \& \quad Q(\varepsilon, t) = 0$$

$$-\frac{d\varepsilon}{dt} = C_{\text{sync}} \varepsilon^2 H^2 \sin^2 \theta \quad \text{where} \quad C_{\text{sync}} = \frac{2e}{3m_e^4 c^7}$$

⇒ It is possible to derive the energy of each particle as a function of time:

$$-\frac{d\varepsilon}{\varepsilon^2} = C_{\text{sync}} H^2 \sin^2 \theta dt \quad \rightarrow \quad \frac{1}{\varepsilon(t)} - \frac{1}{\varepsilon_0} = C_{\text{sync}} H^2 \sin^2 \theta t$$

$$\frac{1}{\varepsilon(t)} = \frac{1}{\varepsilon_0} + C_{\text{sync}} H^2 \sin^2 \theta t = \frac{1 + \varepsilon_0 C_{\text{sync}} H^2 \sin^2 \theta t}{\varepsilon_0}$$

$$\varepsilon(t) = \frac{\varepsilon_0}{1 + \varepsilon_0 C_{\text{sync}} H^2 \sin^2 \theta t}$$

Define the **cooling time t^*** as the ratio between the total particle energy and its loss rate:

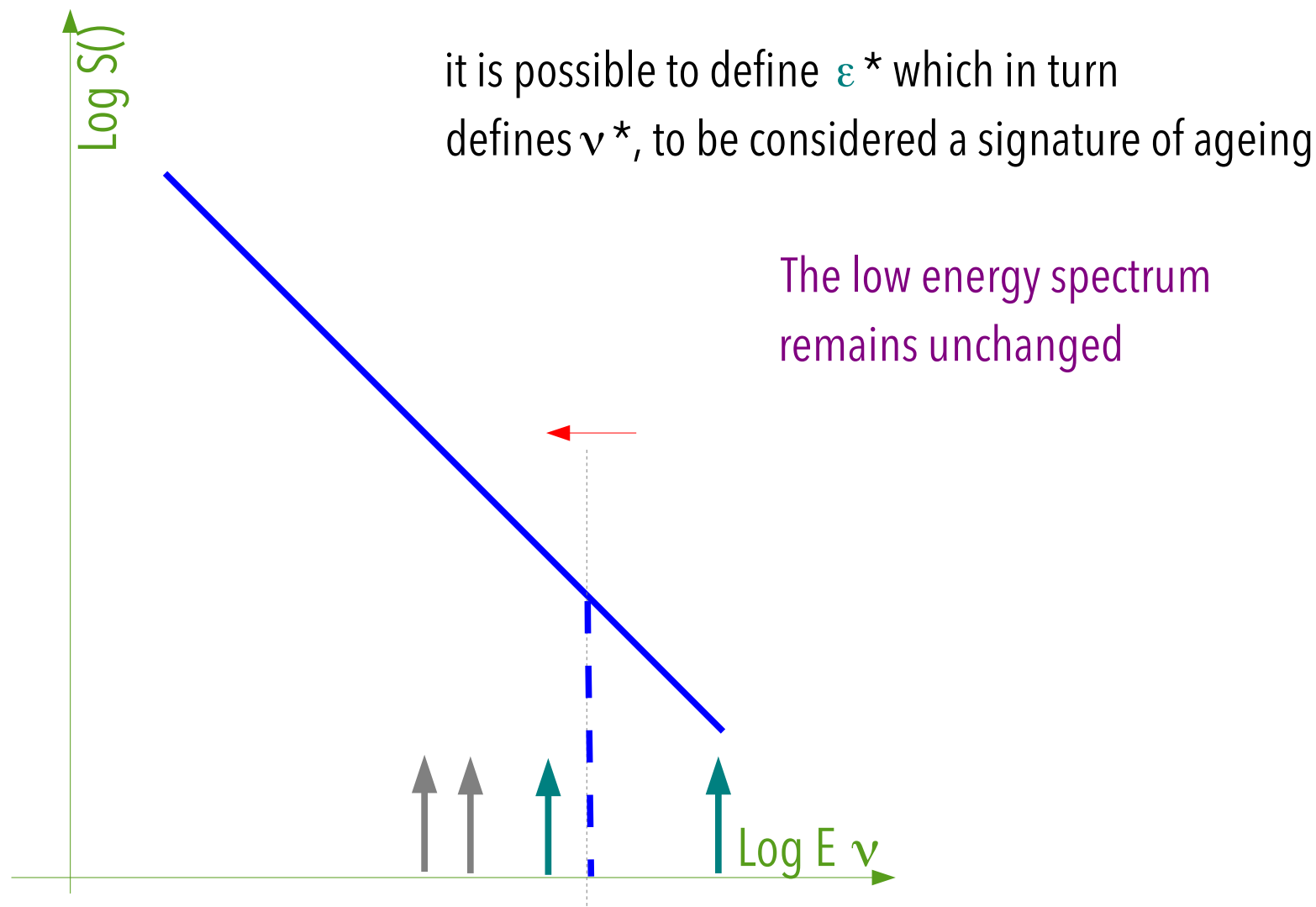
$$t^* = \frac{\varepsilon_0}{d\varepsilon/dt} = \frac{\varepsilon_0}{C_{\text{sync}} \varepsilon_0^2 H^2 \sin^2 \theta} = \frac{1}{C_{\text{sync}} \varepsilon_0 H^2 \sin^2 \theta}$$

With such t^* we can rewrite the particle energy with time

$$\varepsilon(t) = \frac{\varepsilon_0}{1 + (t/t^*)}$$

High energy particles have shorter t^* than low energy ones

The emission spectrum is modified accordingly due to the depletion of the high energy particles from the energy distribution:





Time evolution: A case study (more general)

$$\frac{\partial N(\varepsilon, t)}{\partial t} + \frac{\partial}{\partial \varepsilon} \left(\frac{d\varepsilon}{dt} N(\varepsilon, t) \right) + \frac{N(\varepsilon, t)}{T_{\text{conf}}} = Q(\varepsilon, t)$$

1. The confinement time is extremely large

$$T_{\text{conf}} \rightarrow \infty$$

2. There is continuous injection of fresh electrons:

$$Q(\varepsilon, t) = A \varepsilon^{-\delta}$$

A balance between dying and refurbished particles is achieved at a particular energy, corresponding to a particular frequency, and, in turn, to a characteristic time; one solution of the equation is such that the synchrotron emissivity becomes:

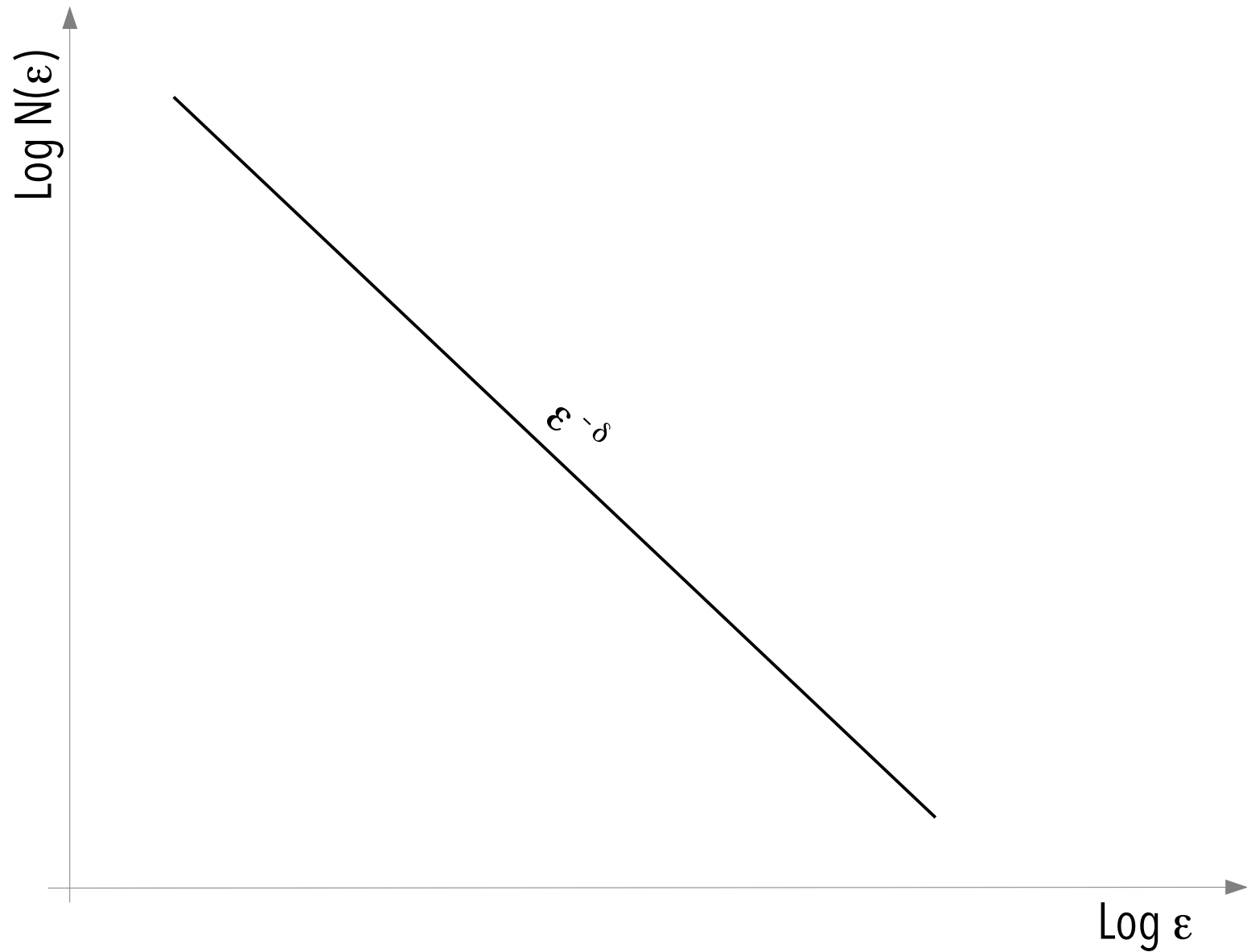
$$J_s(\nu) \approx \nu^{-(\delta-1)/2} = \nu^{-\alpha} \quad \text{when } \nu \ll \nu^*$$

$$J_s(\nu) \approx \nu^{-\delta/2} = \nu^{-(\alpha+1/2)} \quad \text{when } \nu > \nu^*$$



Spectral "ageing": How does it work?

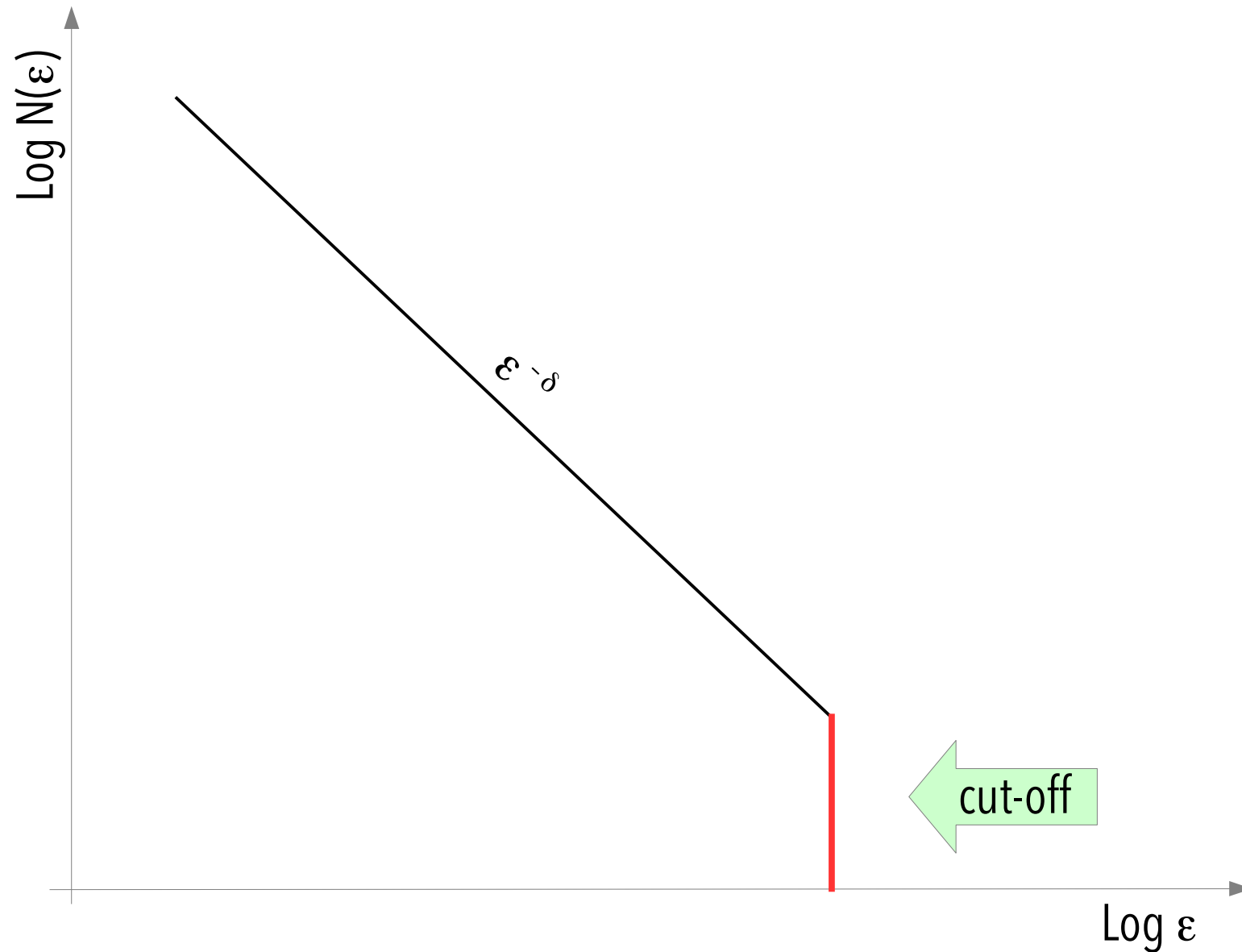
1. Initial power-law distribution



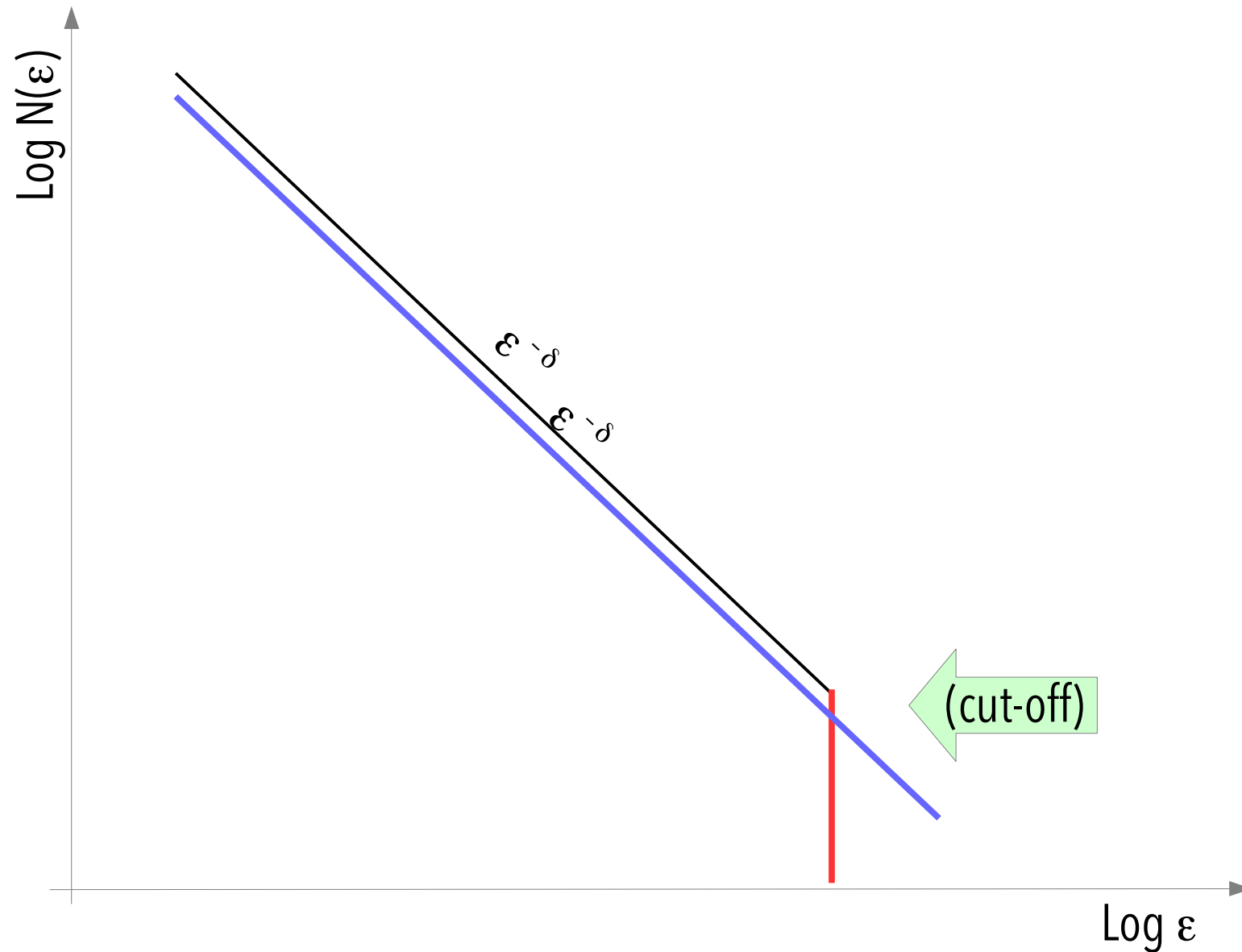


Spectral "ageing": How does it work?

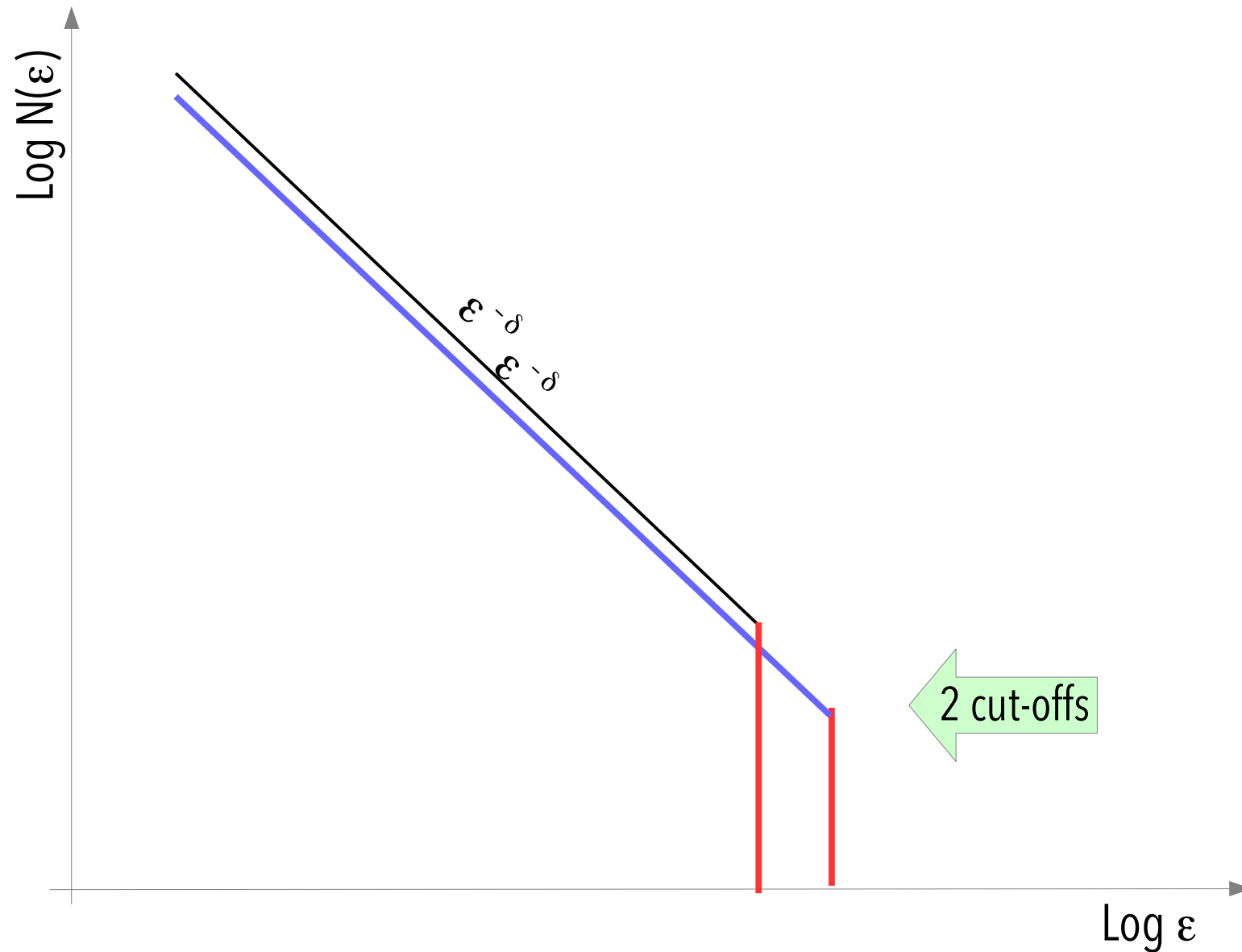
2. Initial power-law distribution depletes high energies as it ages



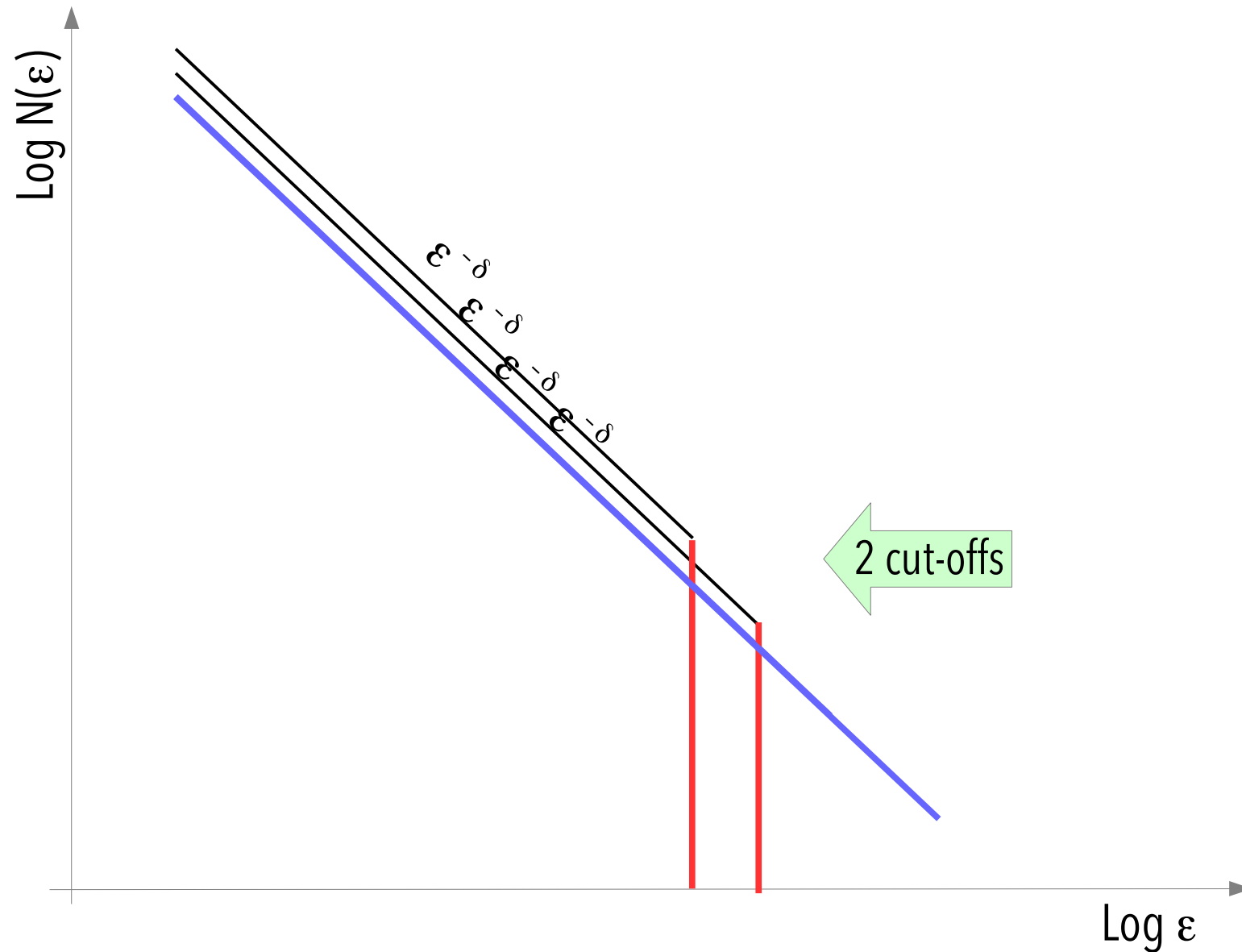
3. Aged initial power-law distribution and new generation of plasma (1)



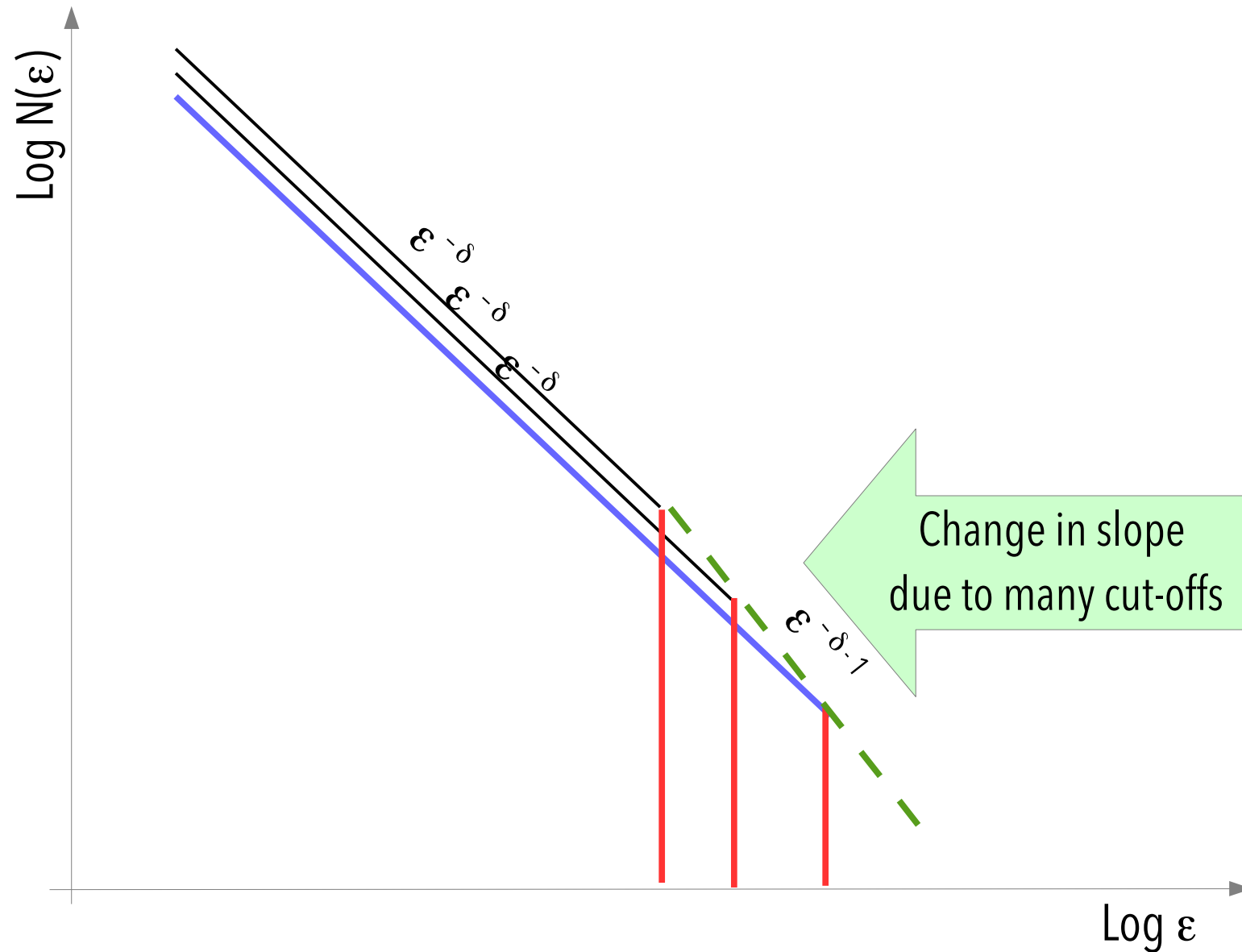
4. Aged initial power-law distribution and (a second) new generation of plasma (2).



5. Aged initial power-law distributions and new (third) generation of plasma

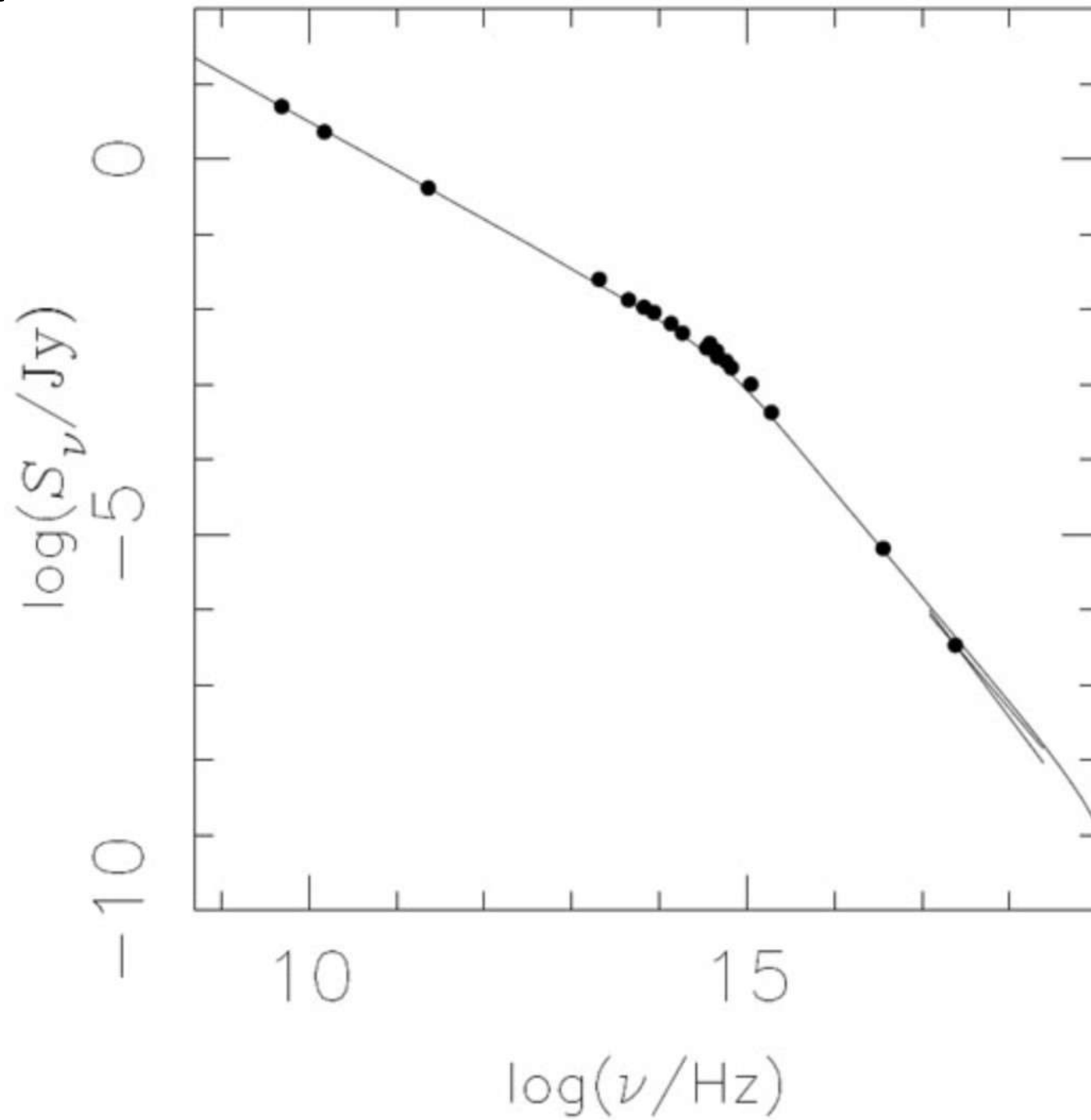


6. Aged initial power-law distributions and aged new generation of plasma



Spectral “ageing”

The integrated spectrum of a radio source



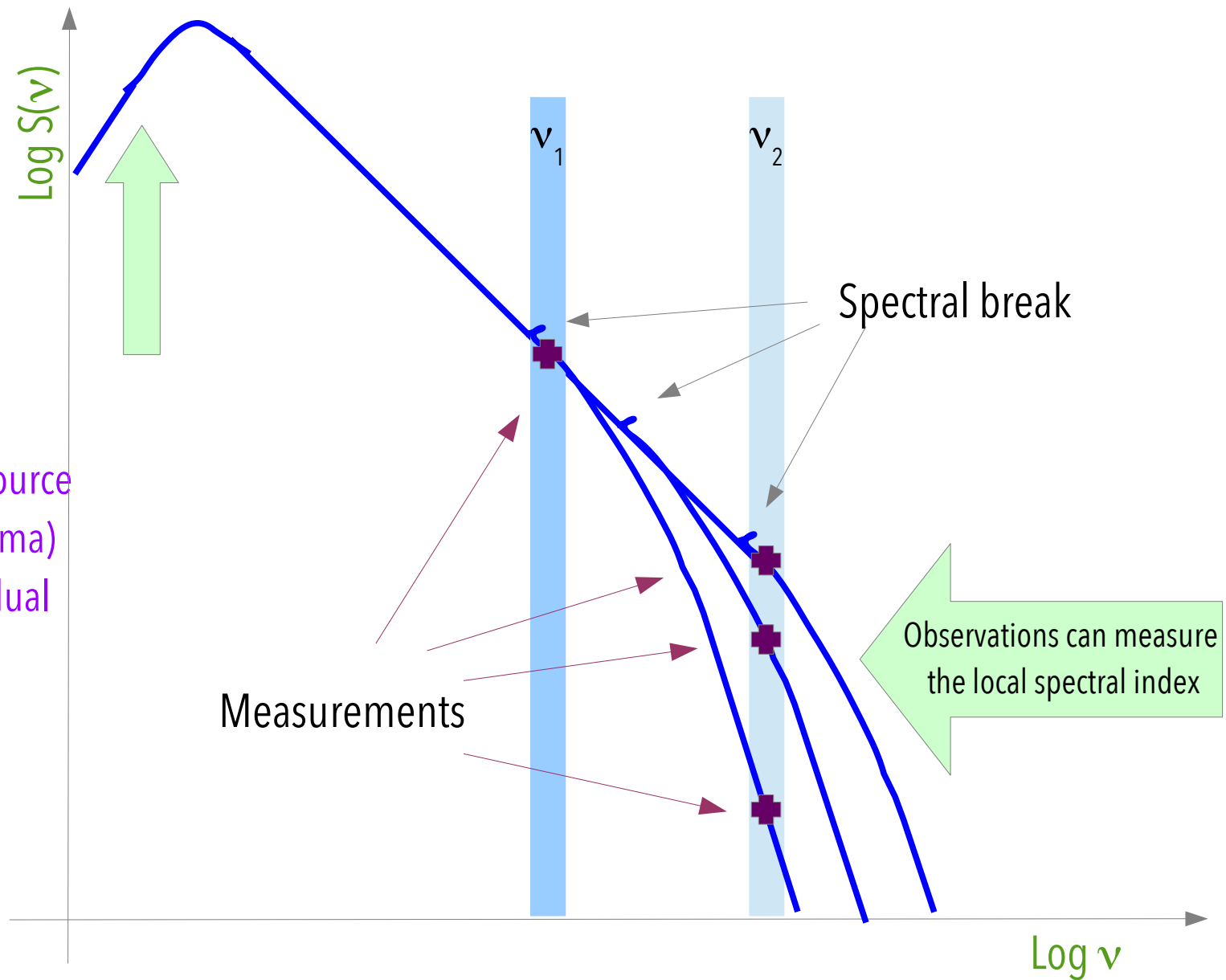


Spectral "ageing"

The search for a break in the high frequency spectrum is a tool to estimate the radiative age of the dominant population of relativistic electrons

Different regions of the same radio source may have different radiative age based on:

1. injection time
2. location within the r-source (i.e. history of the plasma)
3. information on individual radio source evolution





0.5

1

1.5

SPIX

The "age" of electrons is different in various region of a radio source

Info on the radio source formation and evolution

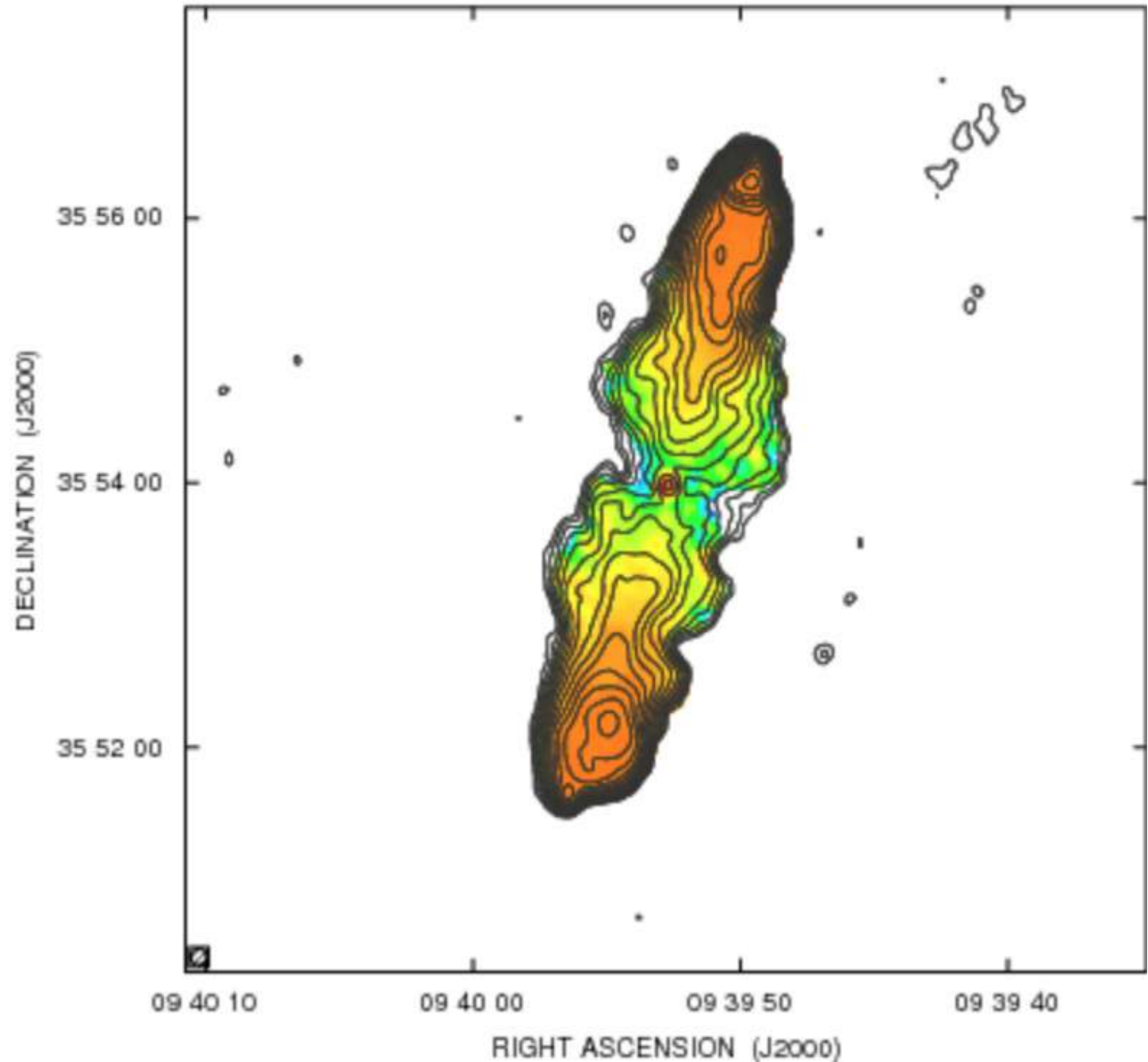
Image of the spectral index

α (color scale)

overimposed to iso-brightness contour levels

$$S(\nu) \approx \nu^{-\alpha}$$

$$\alpha = \frac{\log \frac{S(\nu_2)}{S(\nu_1)}}{\log \frac{\nu_2}{\nu_1}}$$





0.5

1

1.5

SPIX

The "age" of electrons is different in various region of a radio source

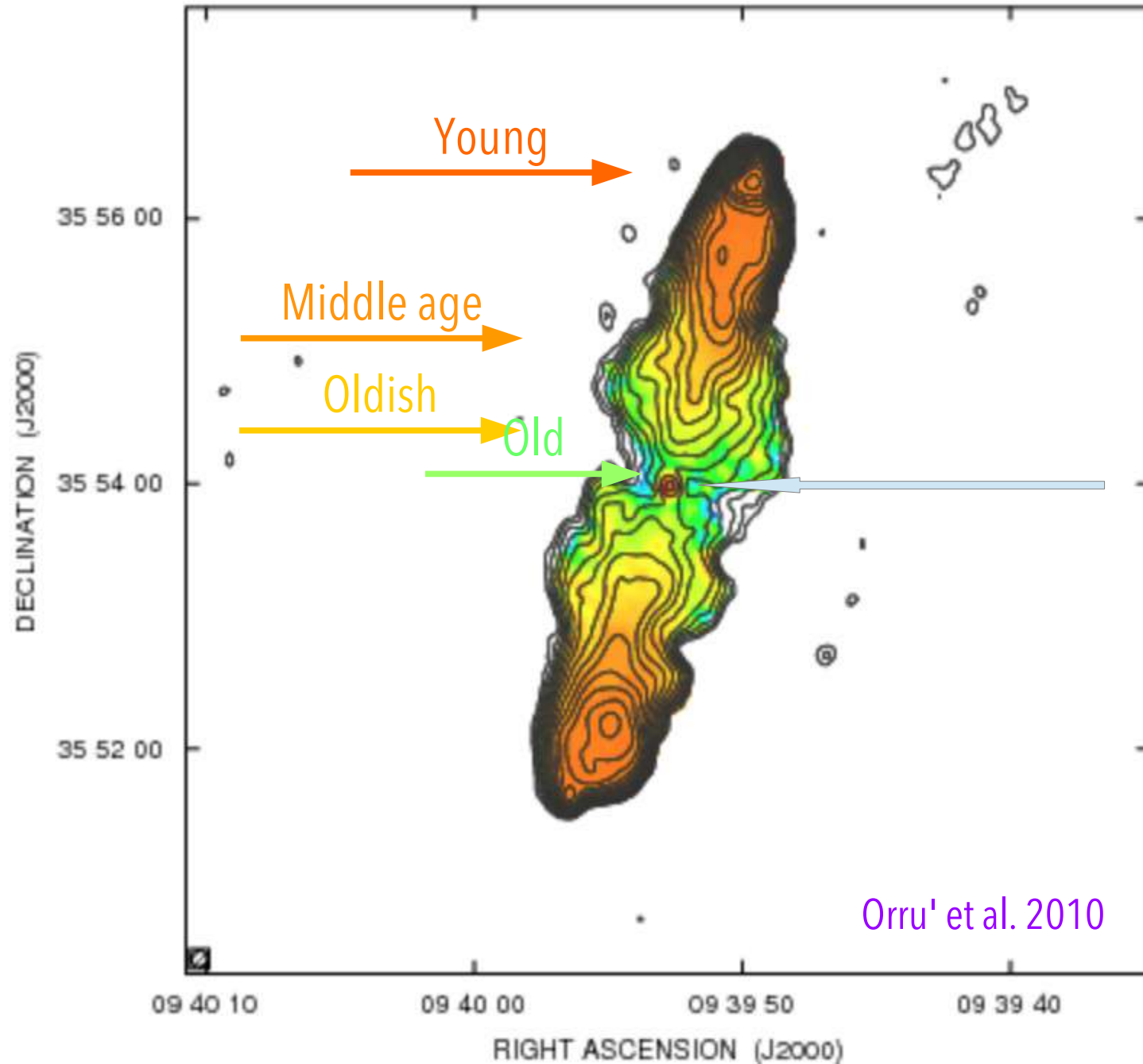
Info on the radio source formation and evolution

Image of the spectral index α (color scale) computed

between two frequencies, overlaid to iso-brightness contour levels

$$S(\nu) \approx \nu^{-\alpha}$$

$$\alpha = \frac{\log \frac{S(\nu_2)}{S(\nu_1)}}{\log \frac{(\nu_2)}{(\nu_1)}}$$

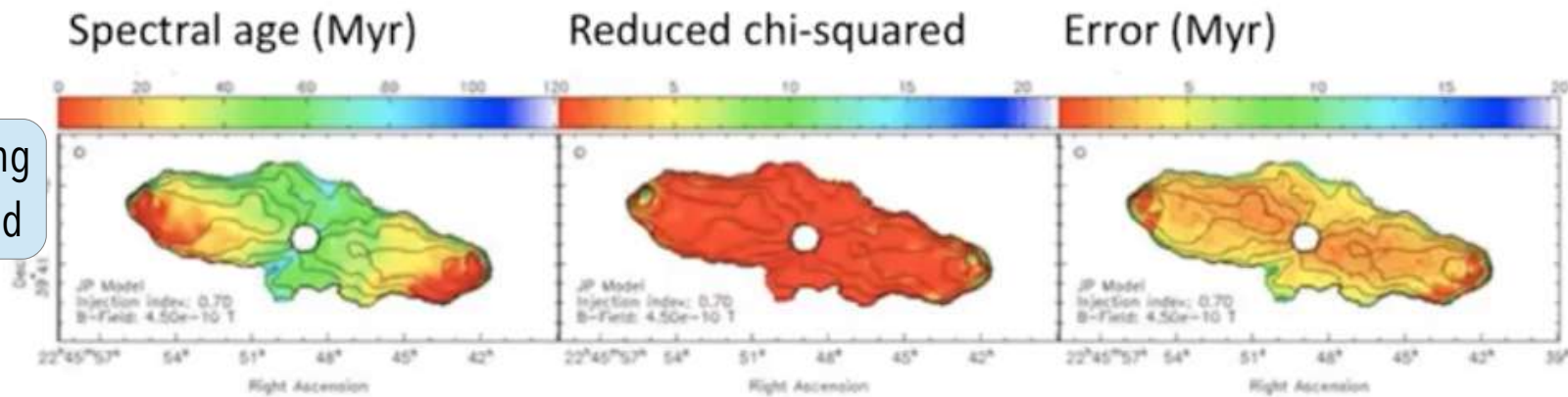


How to measure the age of the e- population

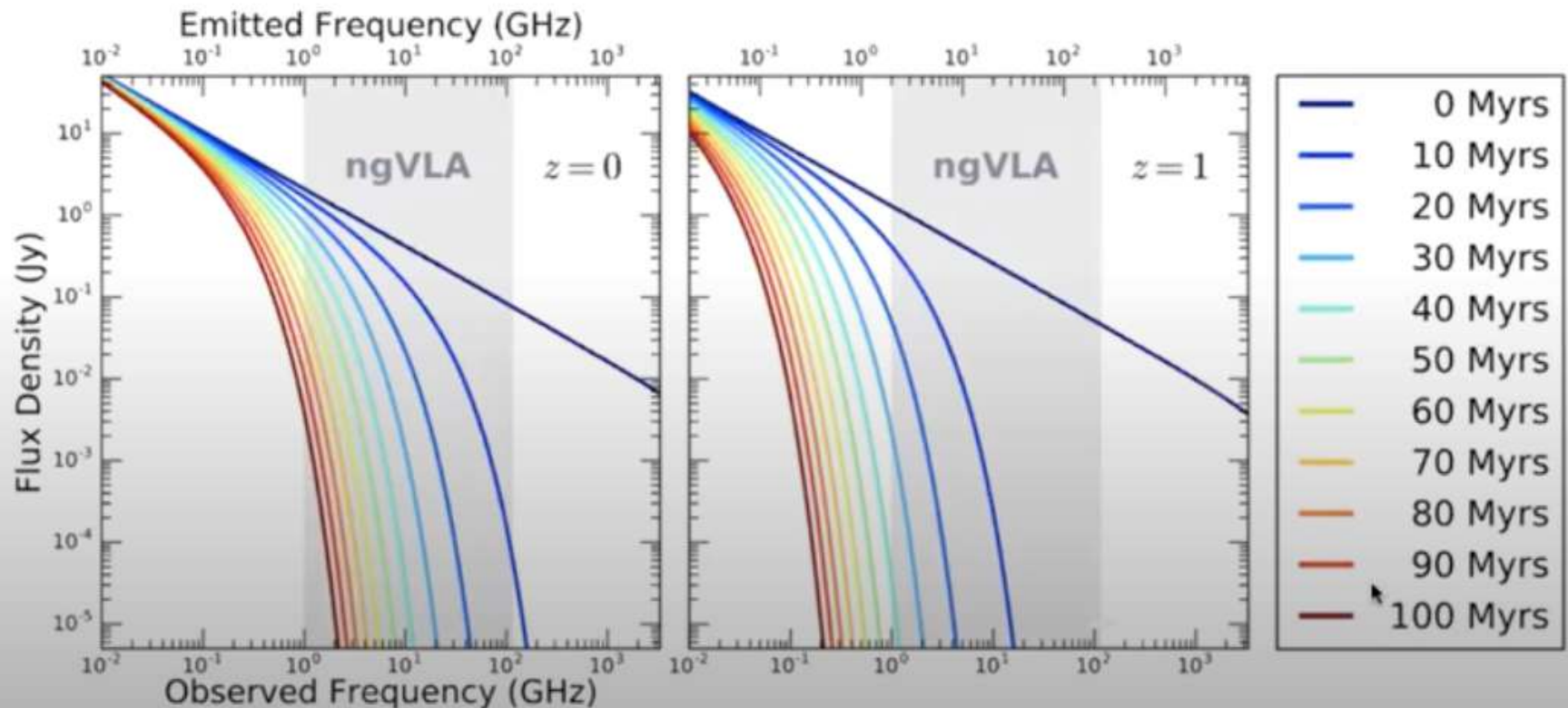
The "age" of electrons is different in various regions of a radio source

Info on the radio source formation and evolution

Red = young
Green = old



Harwood et al. (2017)

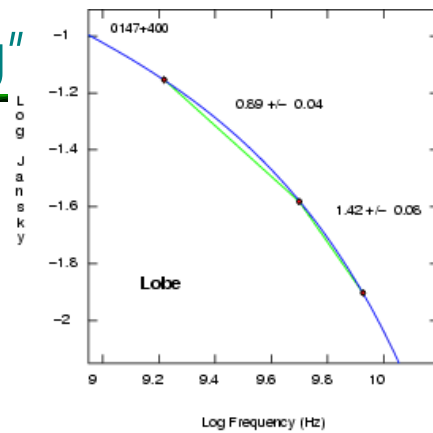
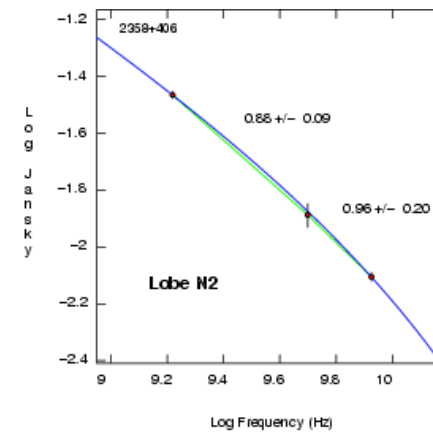
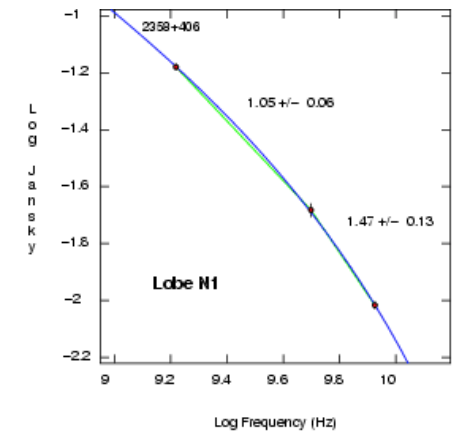
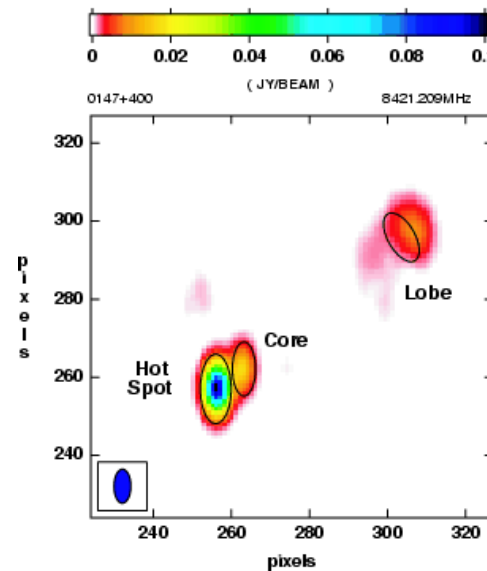
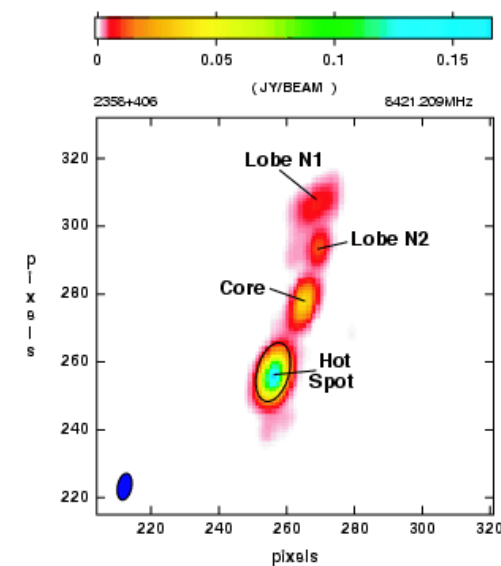
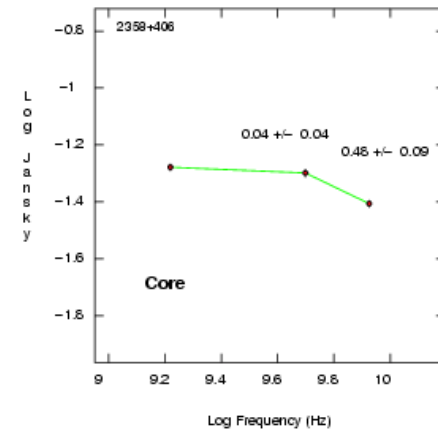
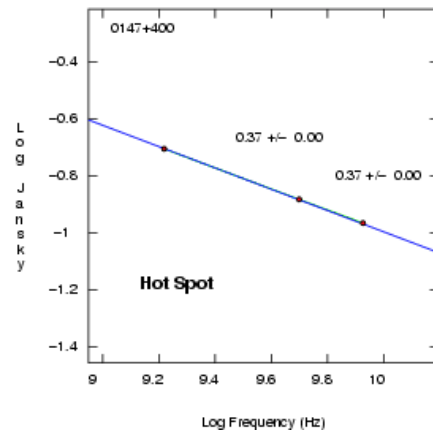
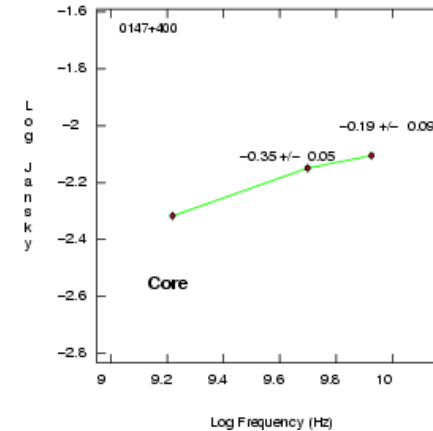
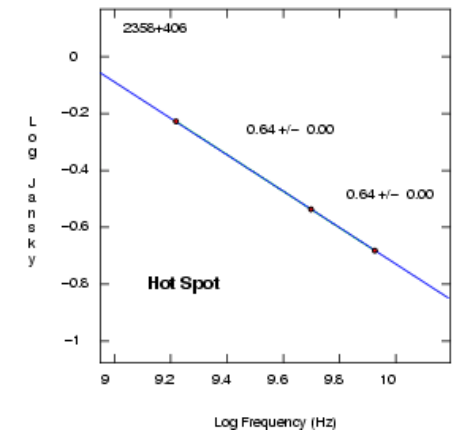


Harwood et al. (2013), Nyland et al. (2018)

Spectral "ageing"

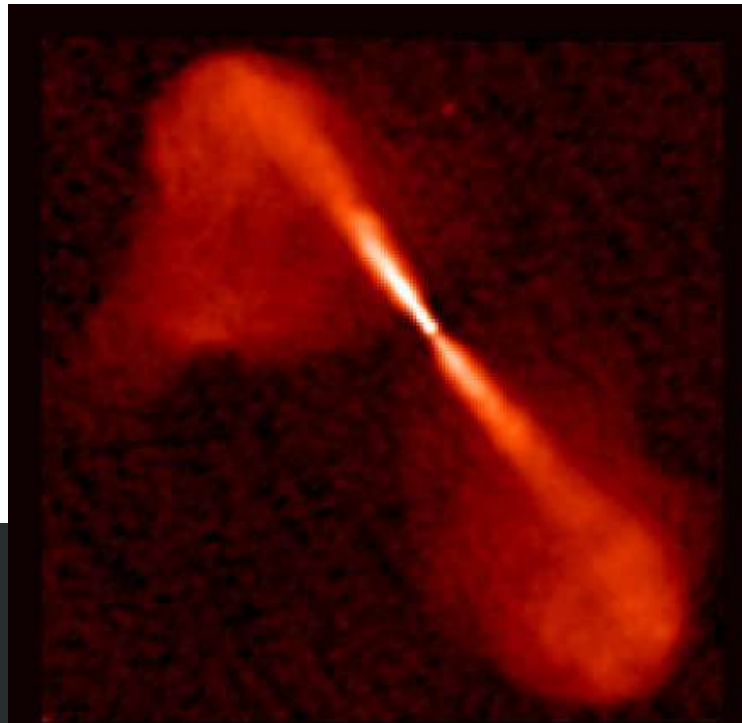
The "age" of electrons is different in various region of a radio source

Info on the radio source formation and evolution

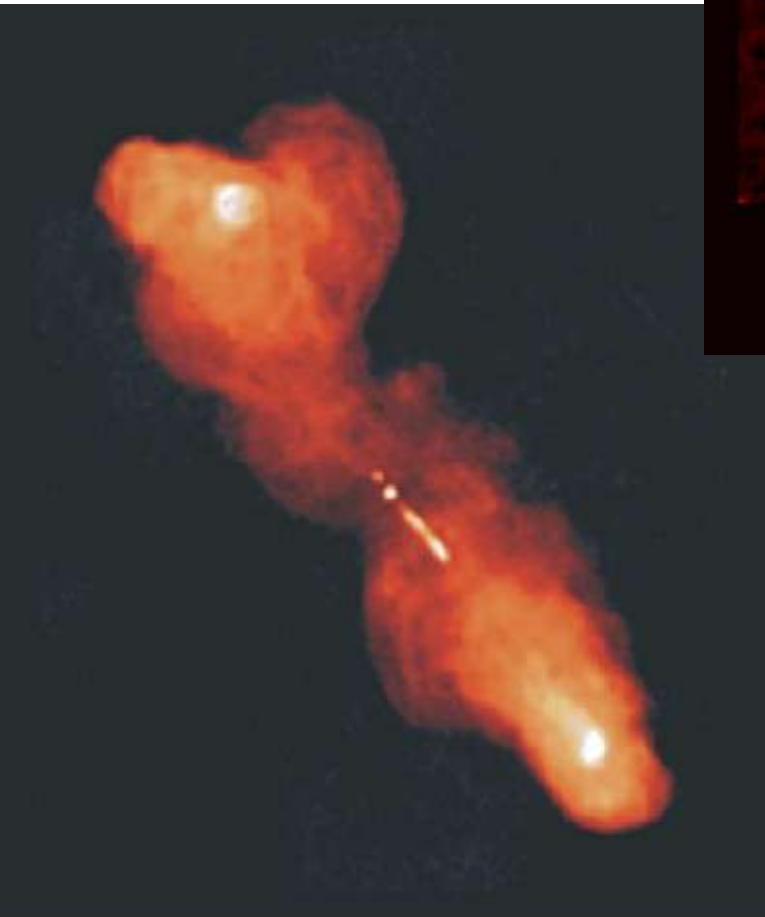
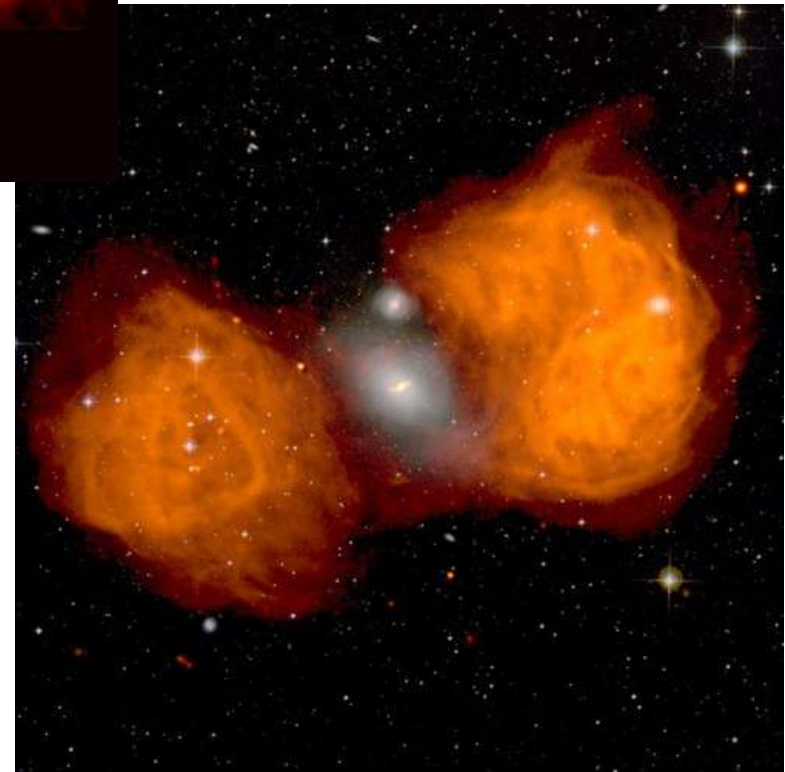




Fresh electrons in an active nuclei & old electrons in dead AGN



Radio Galaxy 3C296
VLA 20cm image
Copyright (c) NRAO/AUI 1999





Adiabatic expansion:

Poisson's law: $TV^{\Gamma-1} = \text{const} \rightarrow Tr^{3(4/3-1)} = Tr = \text{const}$

$$\varepsilon \sim T \sim \frac{1}{r} \quad \varepsilon r = \varepsilon_0 r_0 = \text{const}$$

$$\varepsilon(t) = \varepsilon_0 \frac{r_0}{r} = \varepsilon_0 \frac{r_0}{r_0 + v_{\text{exp}} t}$$

$$\frac{d\varepsilon}{dt} = -\frac{v_{\text{exp}}}{r} \varepsilon$$

FYI,
Computatcions
never asked

$$t_{\text{ad}}^* = \frac{r_0}{v_{\text{exp}}} = \text{const}$$

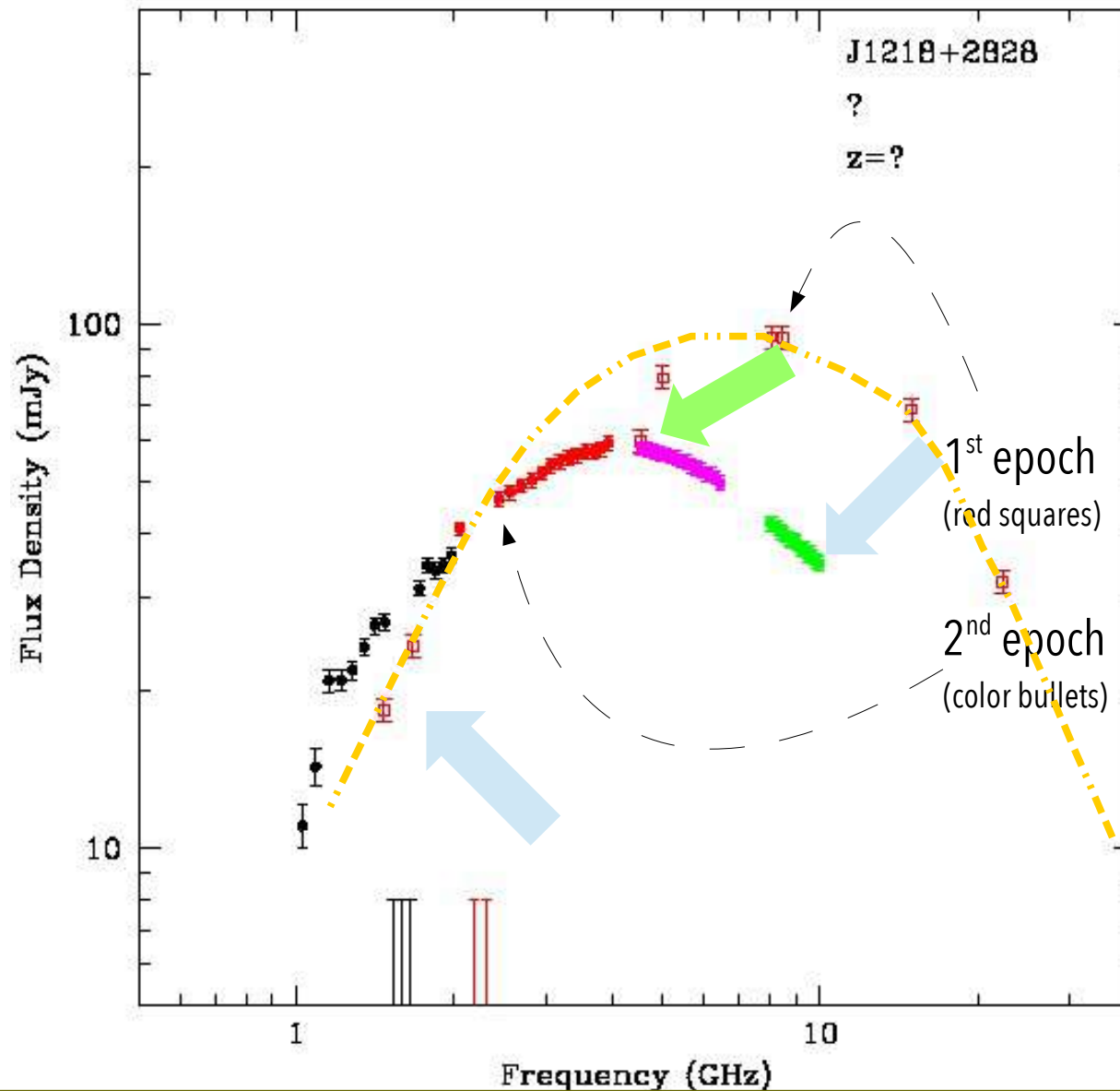
$$\varepsilon(t) = \frac{\varepsilon_0}{1 + t/t_{\text{ad}}^*}$$

magnetic field:

$$H(t) = H_0 \left(\frac{r_0}{r} \right)^2$$

The shape of the spectrum does not change, but it is shifted at lower frequencies

Effects of adiabatic expansion:



The shape of the spectrum does not change, but it is shifted at lower frequencies

The source expansion makes self-absorption less effective and the optically thick

Spectrum gets stronger. The peak in the 2nd epoch spectrum has lower ν_p e weaker S_p

(Linear) Polarization

newline It is the most distinctive property of the synchrotron radiation

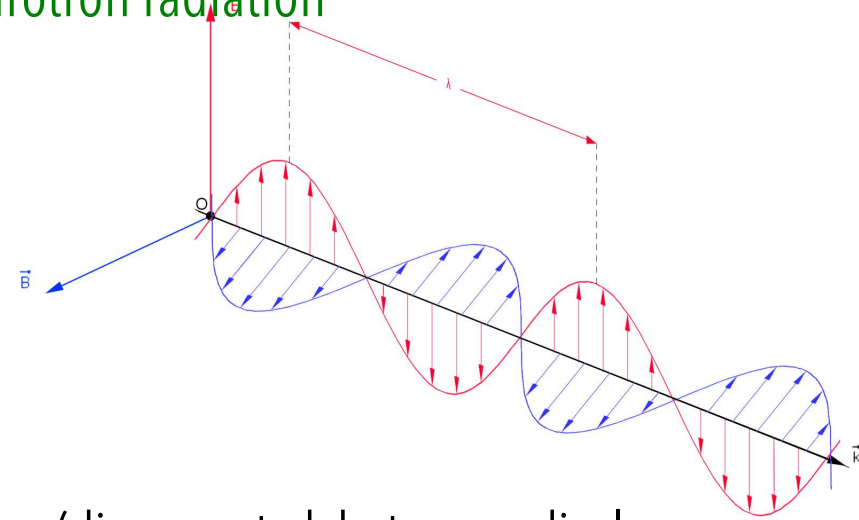
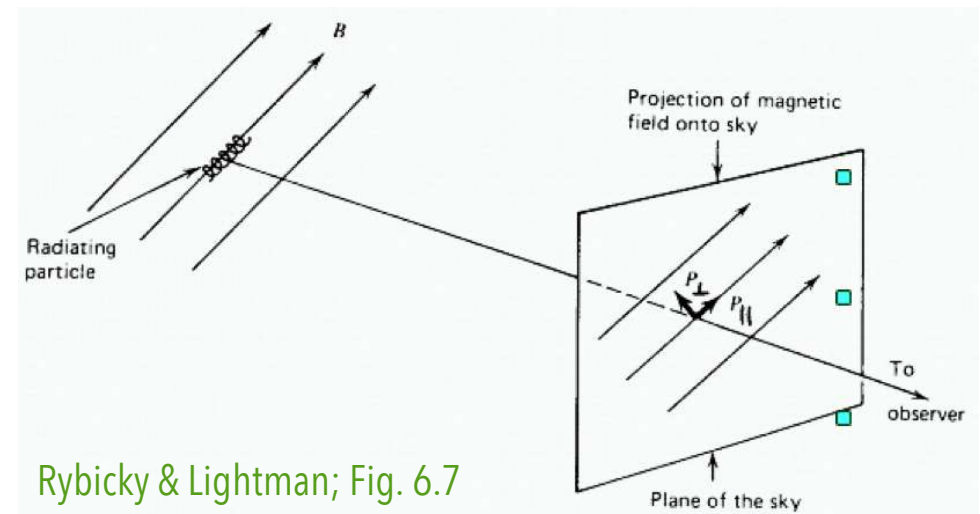
<https://www.youtube.com/watch?v=HH58VmUbOKM>

<https://www.youtube.com/watch?v=8YkfEft4p-w>

↑ For a better understanding

When many photons have E and B fields oscillating in random (disconnected, but perpendicular each other) the radiation is unpolarized

In case radiation has been produced in such a way that, e.g., the E field has a "preferred" direction, then the radiation becomes **linearly polarized**



Rybicky & Lightman; Fig. 6.7

The radiation is emitted with the E field oscillating perpendicularly to the local B field direction

Fractional polarization never reaches 100%
Reduction from $1/\gamma$ opening angle,
and propagation effects

Two components can be separated $P_{\parallel}, P_{\perp}$
Algebra is very complex

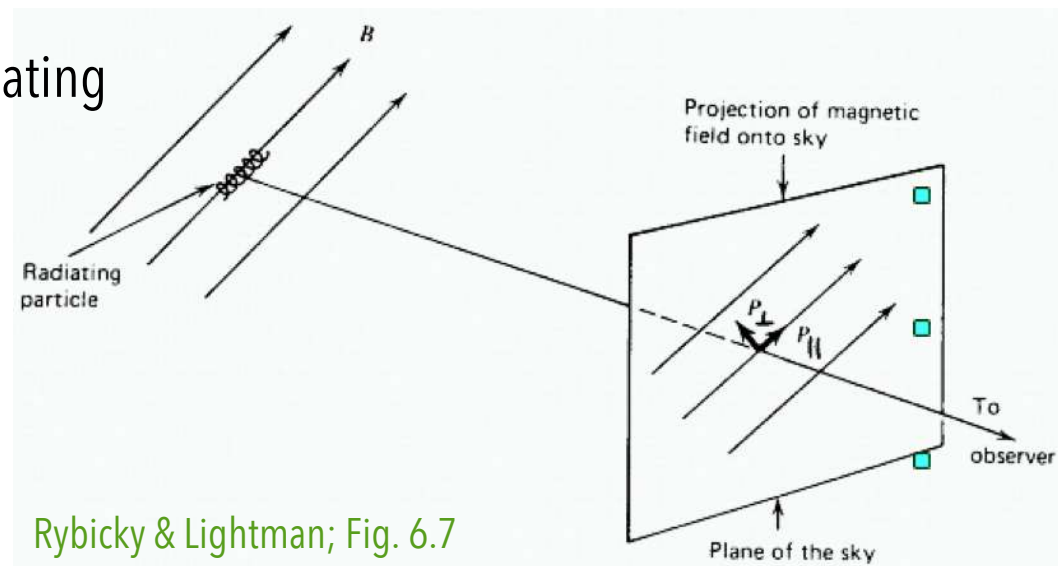
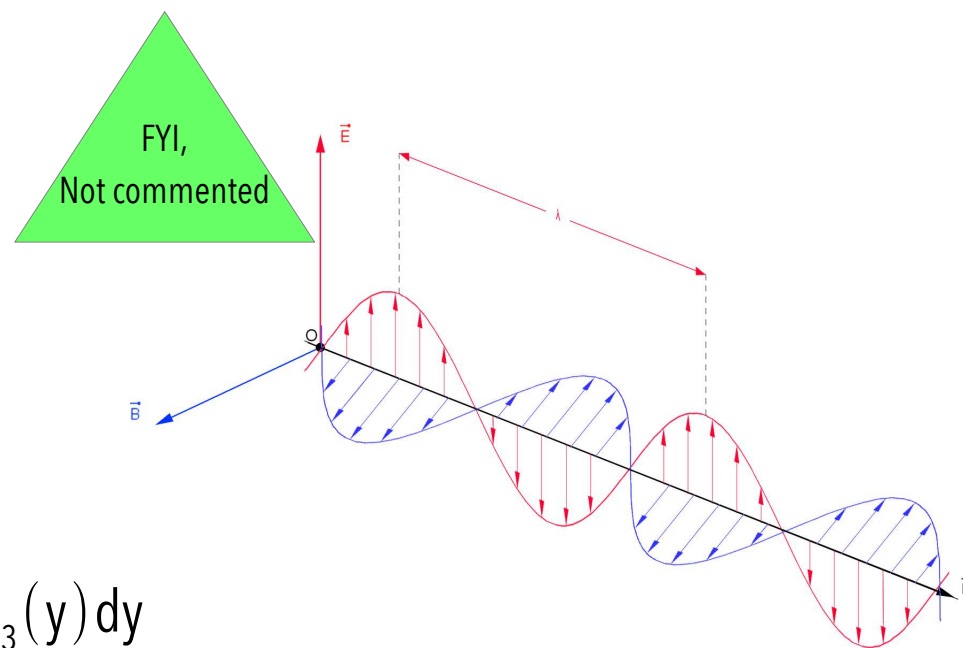
$$\begin{pmatrix} P_{\parallel} \\ P_{\perp} \end{pmatrix} = \frac{\sqrt{3}e^3 B}{2m_e c^2} \begin{pmatrix} F(\nu/\nu_s) - G(\nu/\nu_s) \\ F(\nu/\nu_s) + G(\nu/\nu_s) \end{pmatrix}$$

where

$$F(\nu/\nu_s) = (\nu/\nu_s) \int_{\nu/\nu_s}^{\infty} K_{5/3}(y) dy$$

$$G(\nu/\nu_s) = (\nu/\nu_s) K_{2/3}(\nu/\nu_s)$$

$K \rightarrow$ modified Bessel functions





(Linear) Polarization (3)

The total emitted power for monoenergetic electrons is

$$P(\nu) = P_{\parallel}(\nu) + P_{\perp}(\nu) \approx F(\nu)$$

Integrating over the energies a power-law electron energy distribution we get:

$$\left(\frac{P_{\parallel}(\nu)}{P_{\perp}(\nu)} \right) = \left(\frac{\sqrt{3}}{2} \right) N_0 \frac{e^3 H}{m_e c^2} \left(\frac{J_F - J_G}{J_F + J_G} \right) \left(\frac{2\nu}{3\nu_L} \right)^{-(\delta-1)/2}$$

where

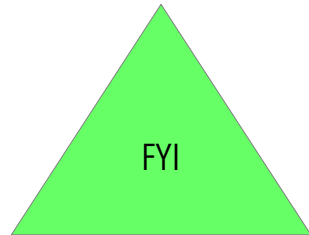
$$J_F = \frac{2^{(\delta+1)/2}}{\delta+1} \Gamma\left(\frac{p}{4} + \frac{19}{12}\right) \Gamma\left(\frac{p}{4} - \frac{19}{12}\right)$$
$$J_G = 2^{(p-3)/2} \Gamma\left(\frac{p}{4} + \frac{7}{12}\right) \Gamma\left(\frac{p}{4} - \frac{1}{12}\right)$$

FYI

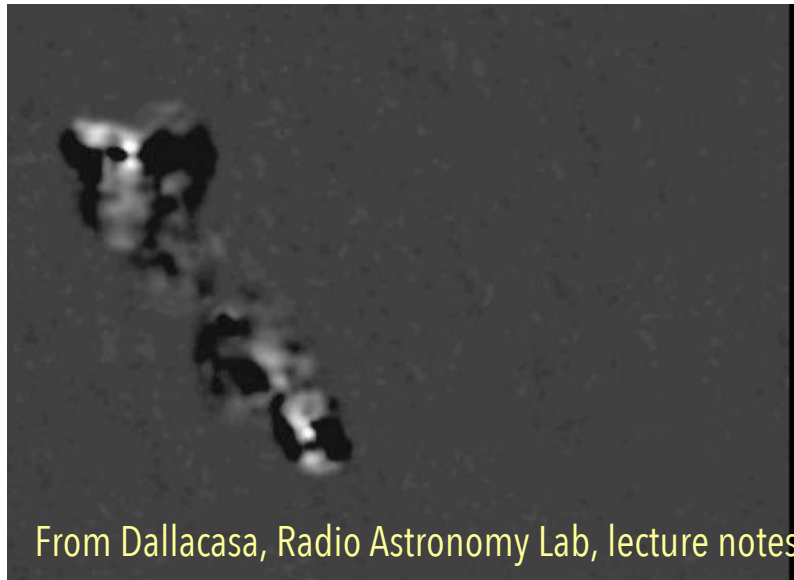
For a relativistic plasma bubble with homogenous field, the degree of polarization is defined by

$$\frac{P_{\perp}(\nu) - P_{\parallel}(\nu)}{P_{\perp}(\nu) + P_{\parallel}(\nu)} = \frac{J_G}{J_F} = \frac{\delta + 1}{\delta + 7/3}$$

The fractional polarization may reach 69.3%, 72.5% and 75.0% in case of $\delta = 2.0, 2.5$ and 3.0 , respectively. This is very large!

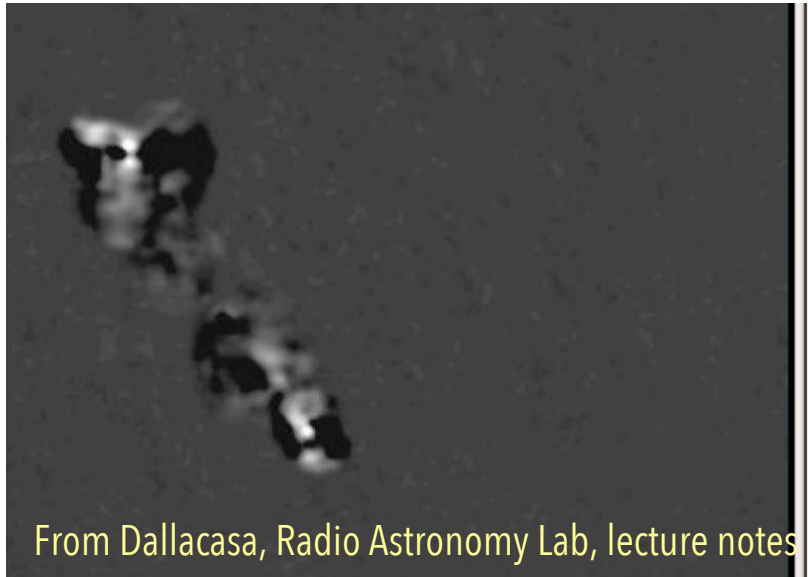
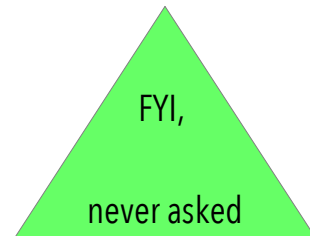
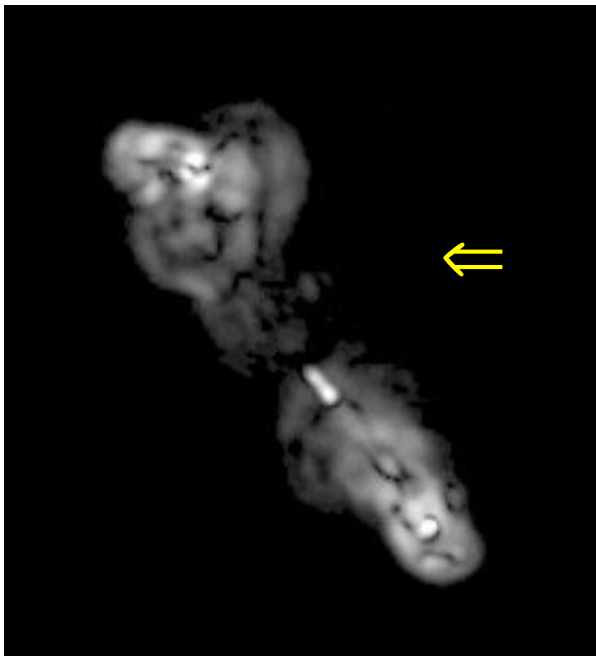


There are effects that may reduce the actual fractional polarization (see further, e.g. inhomogeneous field, differential Faraday Rotation, insufficient resolution, ...)



(Linear) Polarization (5)

Polarization is a measure of the predominant direction of the E field → a vector



For each pixel, the vector length P (side image) is given by

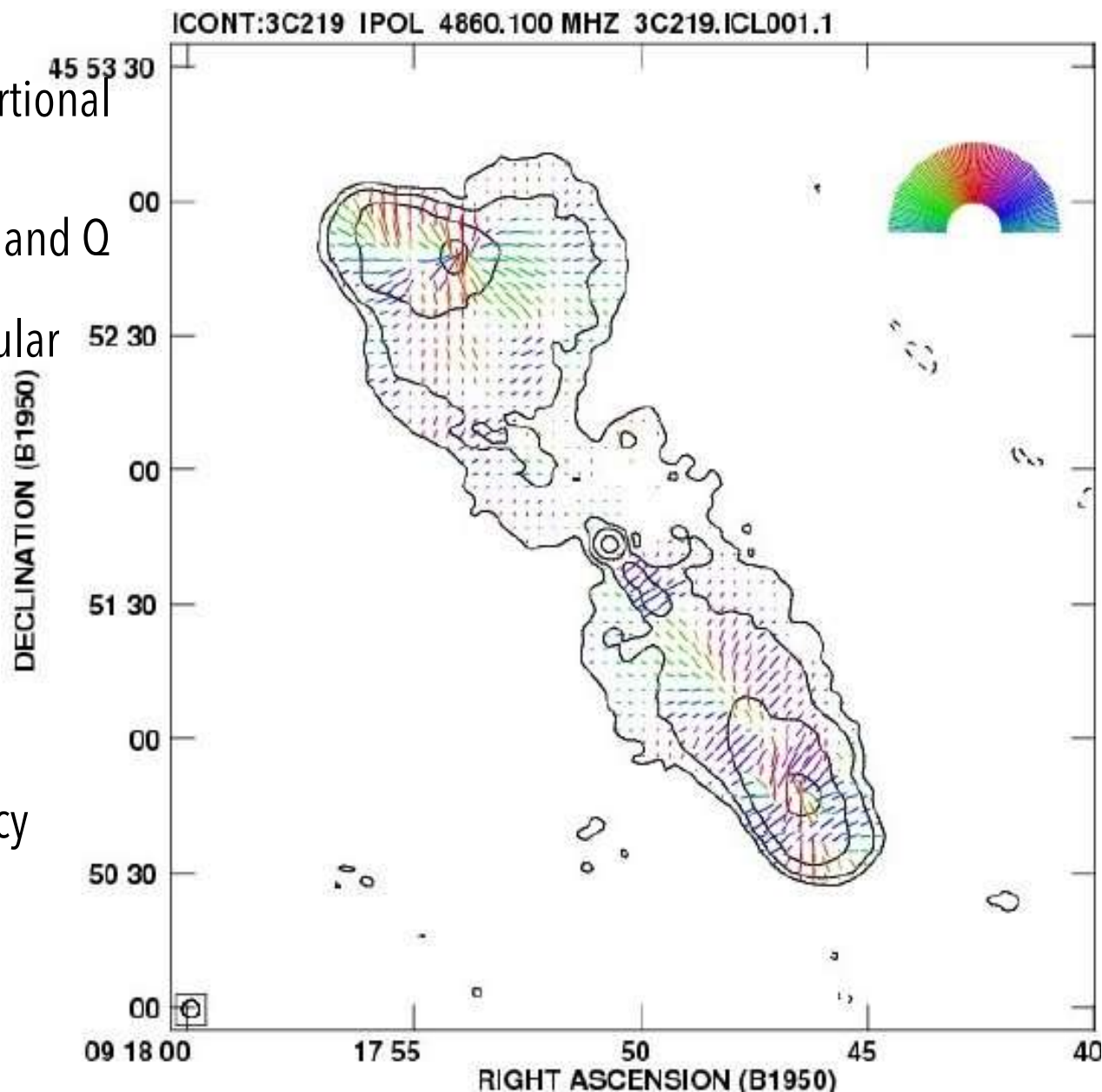
$$P = \sqrt{U^2 + Q^2}$$

For each pixel the orientation of the vector (not shown) is given by

$$\chi = \frac{1}{2} \text{atan}(U, Q)$$

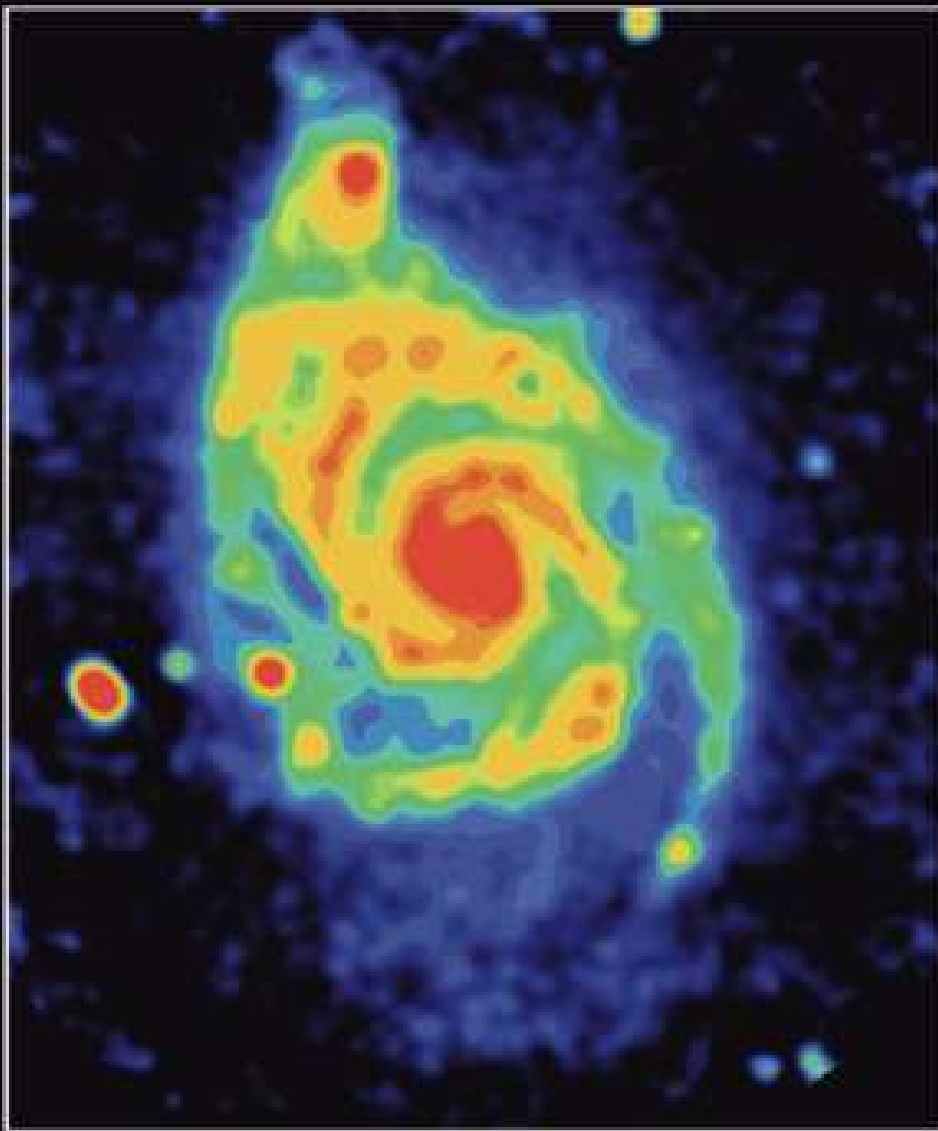
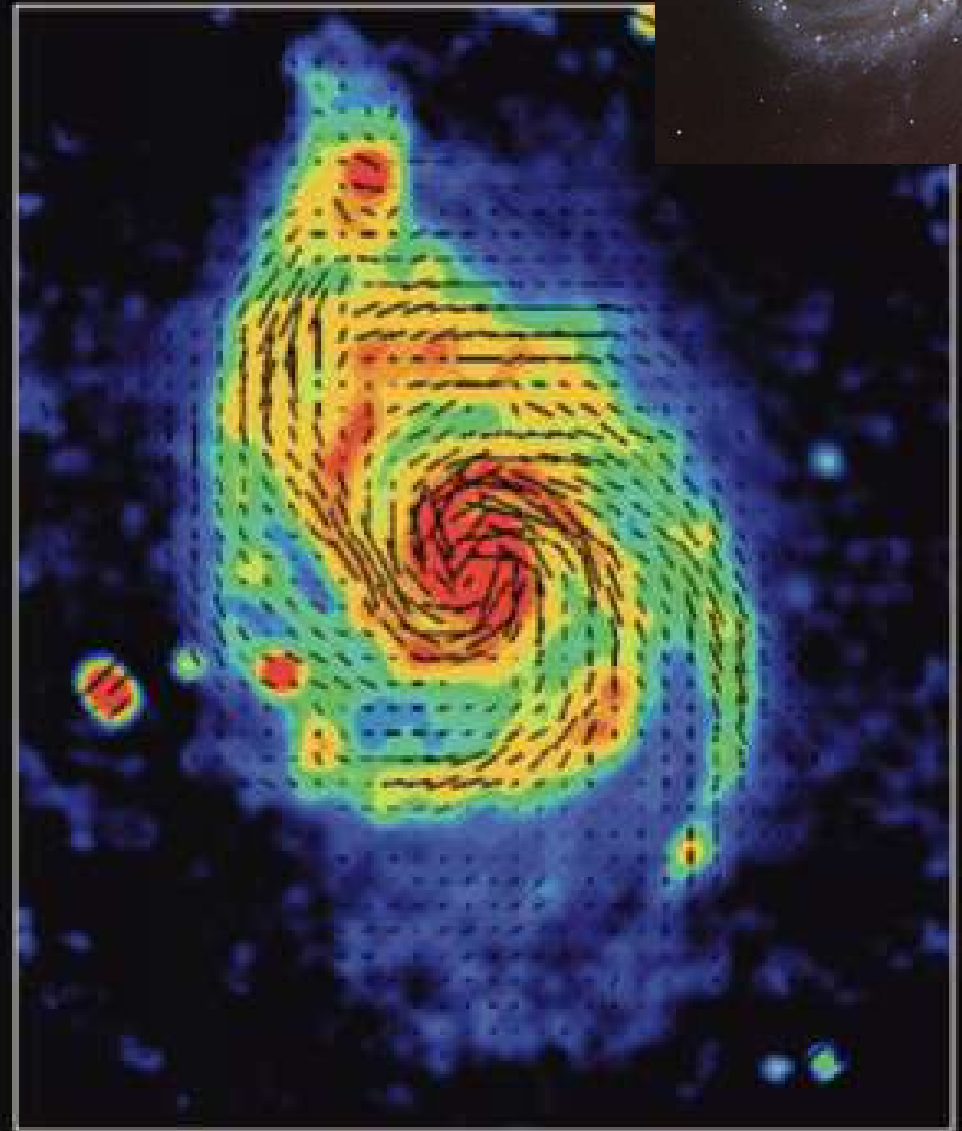
Example 2: polarized emission in the radio galaxy 3C219

- **E-vector displayed**, length proportional to polarization: it has 2 components defined by the Stokes' parameters U and Q
- B-vector (field!) direction perpendicular to the vectors shown here.
- In case resolution is not adequate, the detected polarization emission may be severely reduced (beam depolarization).
- Often polarized emission is frequency dependent.



Example of polarized emission: M51

- vector length proportional to projected field strength
- vector direction (**B field shown here**, but E is measured!)





What are the relevant features of the Synchrotron radiation?

0. Ultra-relativistic electron emission in a H field
 - 1a. Electron emission strongly dependent on ν , power-law, typically in the radio band
 - 1b. Non-thermal plasma, power-law energy distribution
2. Linearly polarized emission
3. Electron lifetime proportional to ε^{-1}
4. Time evolution: cut-off or spectral break (in case of continuous injection of fresh particles)
5. SSA $\Rightarrow$ H field measure
6. Minimum energy and the "equipartition H field"



Questions

- (1) What does it happen to "equipartition" when the relativistic plasma ages?
- (2) How long does a radio source live?
- (3) Are we sure to have the correct answer to (2)?
- (4) What are the key points to find "old" radio sources?
- (5) Do intrinsically small radio sources grow with time?

Let's consider the RF in which the curvature radius has uniform motion ($v_{\parallel} = \text{const}$): the particle "feels" an electric field which is indeed the varying magnetic field, which modifies $v_{\perp}$

$$\vec{\nabla} \times \vec{E}' = -\frac{1}{c} \frac{\partial \vec{H}}{\partial t}$$

For each projected orbit the variation of energy can be written as

$$\Delta \left(\frac{1}{2} m v_{\perp}^2 \right) = \oint q \vec{E}' \cdot d\vec{l} = q \int_S \vec{\nabla} \times \vec{E}' \cdot d\vec{S} = -\frac{q}{c} \int_S \frac{\partial \vec{H}}{\partial t} \cdot d\vec{S}$$

If the magnetic field changes smoothly within a single orbit

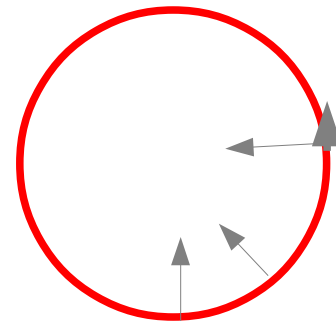
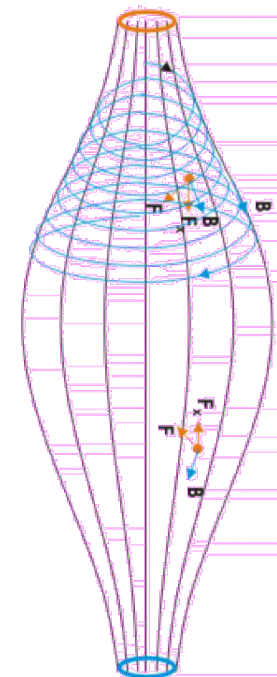
$$\frac{\partial \vec{H}}{\partial t} \approx \frac{\Delta H}{T_L}$$

$$\Delta \varepsilon_{\perp} = \Delta \left(\frac{1}{2} m v_{\perp}^2 \right) = \frac{q}{c} \frac{\Delta H}{T_L} (\pi r_L^2) = \frac{q}{c} \frac{\Delta H}{2\pi r_L} v_{\perp} (\pi r_L^2) \rightarrow$$

$$\text{But } r_L = \frac{\varepsilon}{qH} \rightarrow q \frac{v_{\perp}}{c} \frac{\Delta H}{2} \frac{\varepsilon}{qH} \approx \beta_{\perp} \varepsilon_{\perp} \frac{\Delta H}{H}$$

$$\frac{\Delta \varepsilon_{\perp}}{\varepsilon_{\perp}} \approx \frac{\Delta H}{H} \rightarrow \frac{\varepsilon_{\perp}}{H} = \text{const}$$

FYI





In the observer's frame

- the motion takes place within a static non uniform field;
- the Lorentz force does not make work and the total kinetic energy of the particle does not change with time.

FYI

If $v_{\perp}$ increases (decreases) going into stronger (fainter) H regions, then $v_{\parallel}$ must decrease (increase).

If we consider the energy (scalar), if $\varepsilon_{\perp}$ increases (decreases) then $\varepsilon_{\parallel}$ must decrease (increase) of the same amount

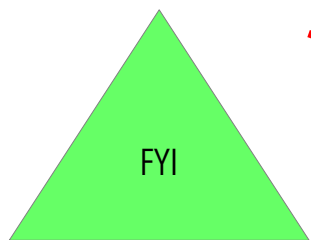
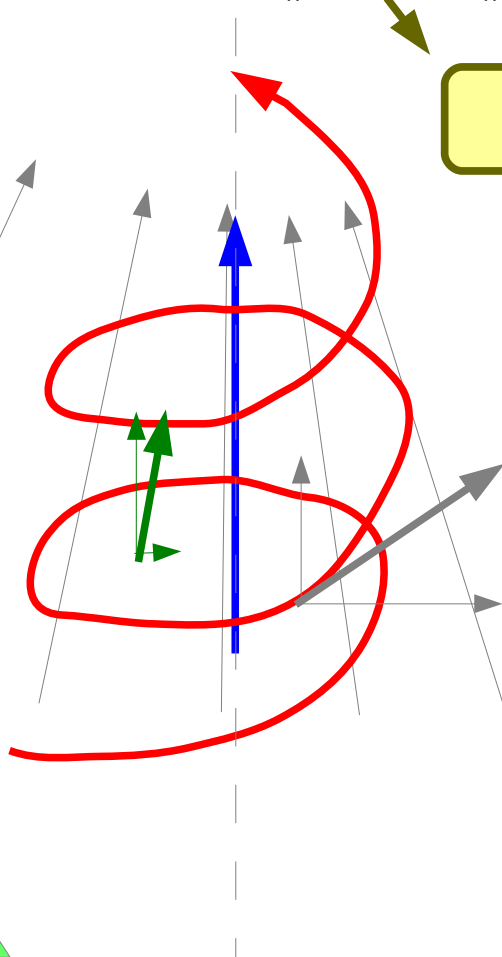
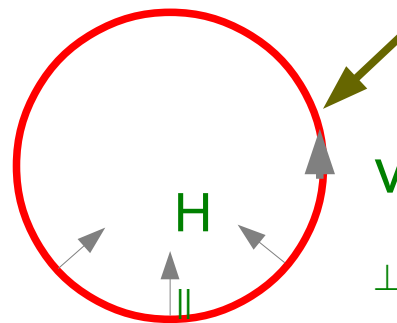


Let's consider velocity and field along and orthogonally to the mean H direction axis.
The motion is within a static field, the Lorentz force does not make work and the total kinetic energy of the particle does not change:

$$\vec{v} \times \vec{H} = (v_{\parallel} + v_{\perp}) \times (H_{\parallel} + H_{\perp}) =$$

$$= v_{\parallel} \times H_{\parallel} + v_{\perp} \times H_{\parallel} + v_{\parallel} \times H_{\perp} + v_{\perp} \times H_{\perp}$$

0

centripetal
accelerationtangent to crf,
(in/de)crease $v_{\perp}$ as Halong helix,
against $v_{\parallel}$ responsible for
reversal
of versus of V



Magnetic mirrors

$$\frac{\Delta \varepsilon_{\perp}}{\varepsilon_{\perp}} \approx \frac{\Delta H}{H} \rightarrow \frac{\varepsilon_{\perp}}{H} = \text{const}$$

It is equivalent to:
$$\frac{\sin^2 \theta}{H} = \text{const}$$

if H_0 and θ_0 (pitch angle) are known at a given point on the particle trajectory, then:

$$\sin \theta = \sin \theta_0 \sqrt{\frac{H}{H_0}}$$

If H increases, then also the pitch angle increases, up to reach the maximum value of 90° and cannot penetrate further regions where H is stronger.

It is then induced to move in the reverse direction

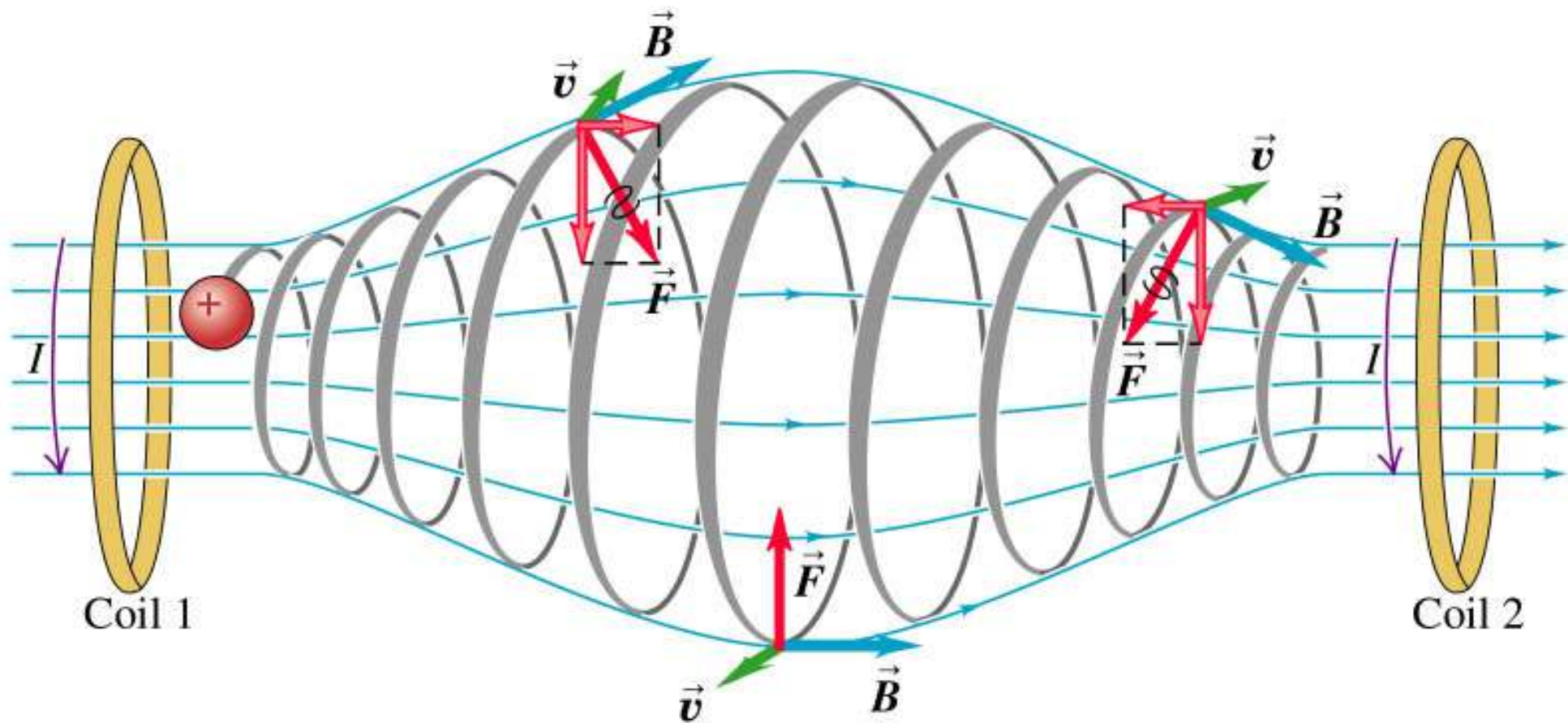
Charged particles may be trapped into regions where the magnetic field is strong enough (example, terrestrial magnetic field)

FYI



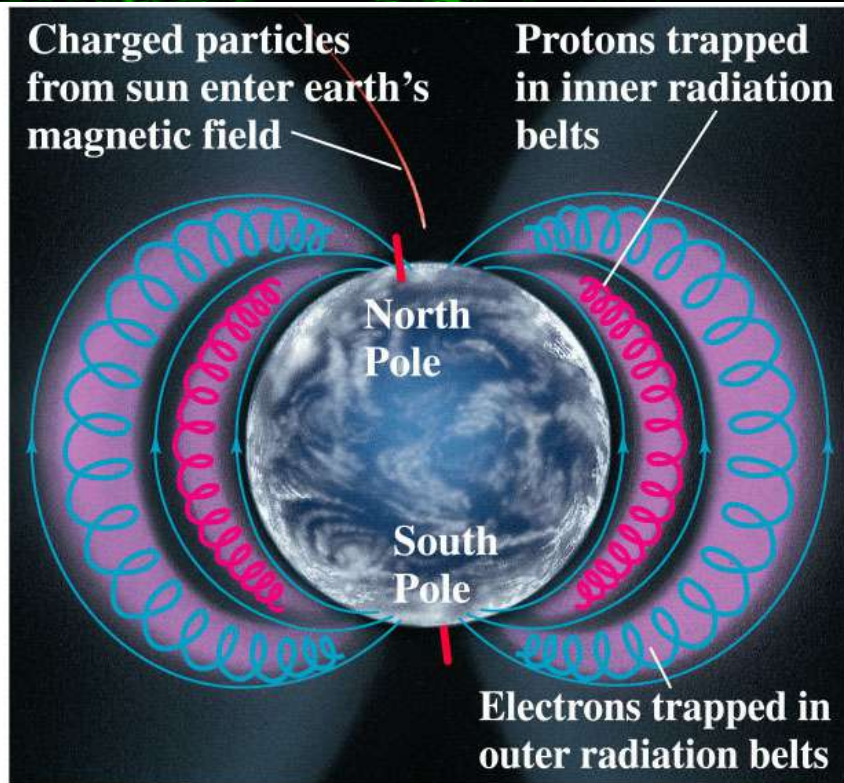
Magnetic trap (bottle)

FYI





Example: Earth Magnetic Field and charged particles



FYI



If a particle is initially very energetic, the magnetic bottle will not be able to confine the particle and it will escape. In exactly the same way, the aurora borealis/australis (Northern and Southern Lights) occurs when charged particles escape from the Van Allen radiation belt. They interact with the atmosphere through optical excitations of the gaseous atoms. Auroras are produced when the magnetosphere is sufficiently disturbed by the solar wind that the trajectories of charged particles in both solar wind and magnetospheric plasma, mainly in the form of electrons and protons, precipitate them into the upper atmosphere (thermosphere/exosphere) due to Earth's magnetic field, where their energy is lost. Auroras form from the interaction of high energy particles (mostly electrons) with neutral atoms in the high atmosphere. These particles may excite (via collisions) bound electrons in the neutral atom. After a characteristic time the excited electrons decay to their initial state, producing radiation. Such process is similar to that in the neon bulbs. The various colors are determined by the composition of the atmosphere and by the colliding particles energy. Atomic oxygen is generally responsible for the red (630 nm) and green (558 nm) light, while molecular nitrogen (both neutral and ionized) causes blue (428 nm) light.

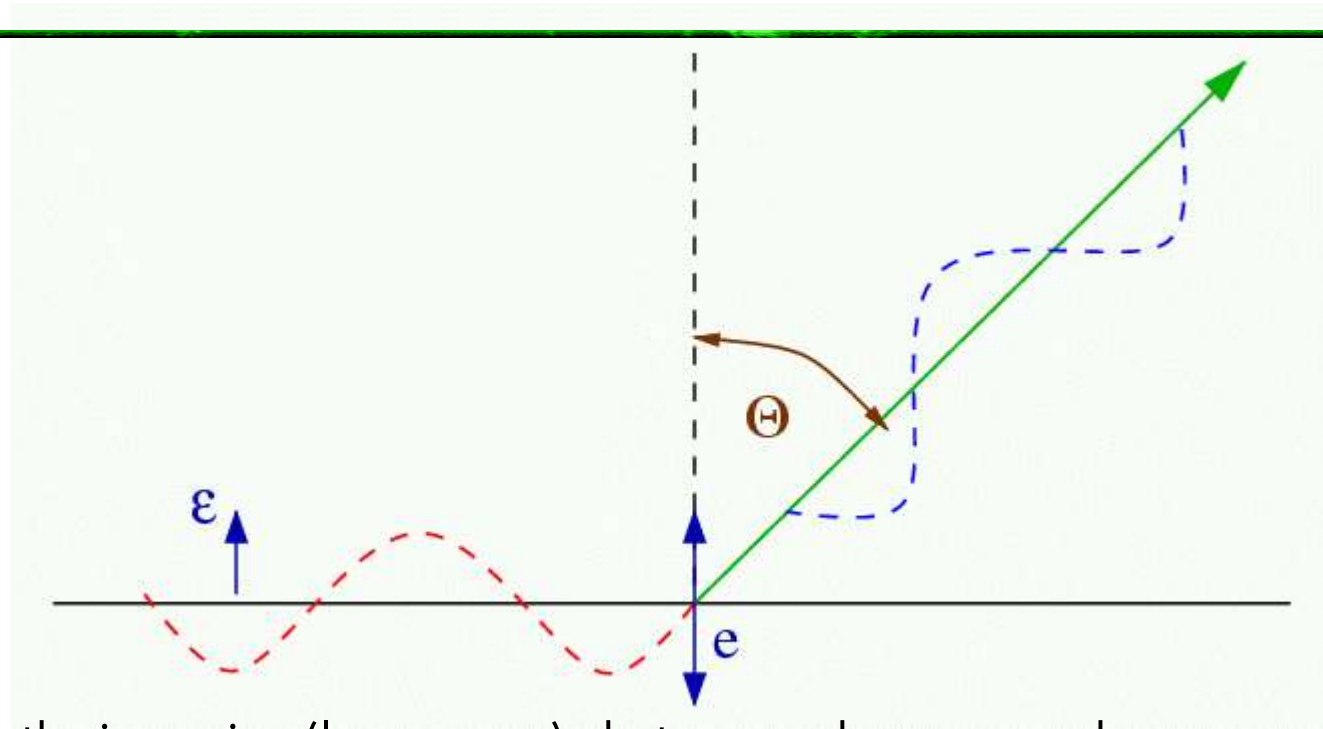


Thompson, Compton & IC scattering on textbooks

- G. Ghisellini: "Radiative Processes in High Energy Astrophysics" § 5
- M. Longair: "High Energy Astrophysics" § 9
- G. B. Rybicky & A.P. Lightman: "Radiative Processes in Astrophysics" § 3.4 § 7

Photon-electron interaction -1-

$$h\nu \ll m_e c^2$$



- for a non relativistic particle, the incoming (low-energy) photons can be expressed as a continuous e-m wave and the magnetic field ($H_0 = E_0$) can be “neglected” (i.e. studied as a consequence of the E field behaviour).
- the average incoming Poynting flux is $\langle |S| \rangle = cE_0^2/8\pi$
- the charge feels a force due to a linearly polarized incoming wave:
- the energy of the scattered radiation is the same as the incoming wave



Thomson scattering (2)

Force from a linearly polarised wave $\vec{F} = e\hat{\epsilon}E_0 \sin \omega_0 t$

Dipole moment

$$\vec{d} = e\vec{r} \quad \rightarrow \quad \ddot{\vec{d}} = e\ddot{\vec{r}} = \frac{e^2 E_0}{m} \hat{\epsilon} \sin \omega_0 t$$

$$\vec{d} = - \left(\frac{e^2 E_0}{m \omega_0^2} \right) \hat{\epsilon} \sin \omega_0 t = \vec{a}(t)$$

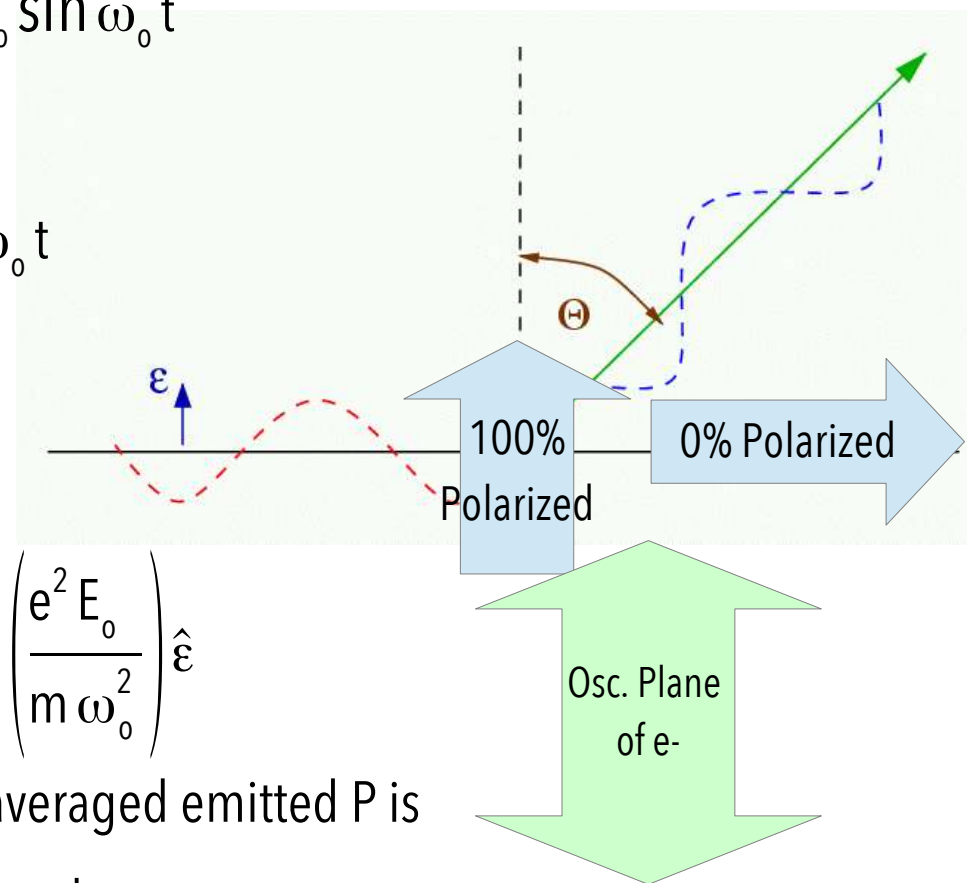
describes a dipole with amplitude $\vec{d}_0 = - \left(\frac{e^2 E_0}{m \omega_0^2} \right) \hat{\epsilon}$

taking into account the Larmor's formula, the time averaged emitted P is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \left\langle \frac{2}{3} \frac{\ddot{\vec{d}}^2}{c^3} \right\rangle = \left\langle \frac{e^4 E_0^2}{8\pi m^2 c^3} \sin^2 \Theta \right\rangle = \langle S \rangle \frac{d\sigma}{d\Omega}$$

where $\frac{d\sigma}{d\Omega} = \frac{e^4}{m^2 c^4} \sin^2 \Theta = r_0^2 \sin^2 \Theta$

Differential cross section for the scattering





An electron oscillates when interacting with the E field of incoming radiation and

$$x(t) = \frac{eE_0}{m_e \omega^2} \cos \omega t$$

the consequent acceleration is $a(t) = \frac{eE_0}{m_e} \cos \omega t$

the emitted power by the scattering particle is $-\left\langle \frac{d\varepsilon}{dt} \right\rangle = \frac{2}{3} \frac{e^2}{c^3} a^2 = \frac{2}{3} \frac{e^2}{c^3} \left(\frac{eE_0}{m_e} \right)^2 \frac{1}{2} = \frac{1}{3} \frac{e^4 E_0^2}{m_e^2 c^3}$

However, remember the module of the Poynting vector of the incoming radiation $\langle |S| \rangle = cE_0^2/8\pi$

the **Thomson cross section** of the electron can be defined:

$$\sigma_T = \frac{\langle d\varepsilon/dt \rangle}{\langle |S| \rangle} = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4} = \frac{8\pi}{3} \left(\frac{e^2}{m_e c^2} \right)^2 = 6.65 \times 10^{-25} \text{ cm}^2$$

where $r_o = \frac{e^2}{m_e c^2} = 2.82 \times 10^{-13} \text{ cm}$ electron classical radius

N.B.: the cross section is independent of frequency (at least apparently)

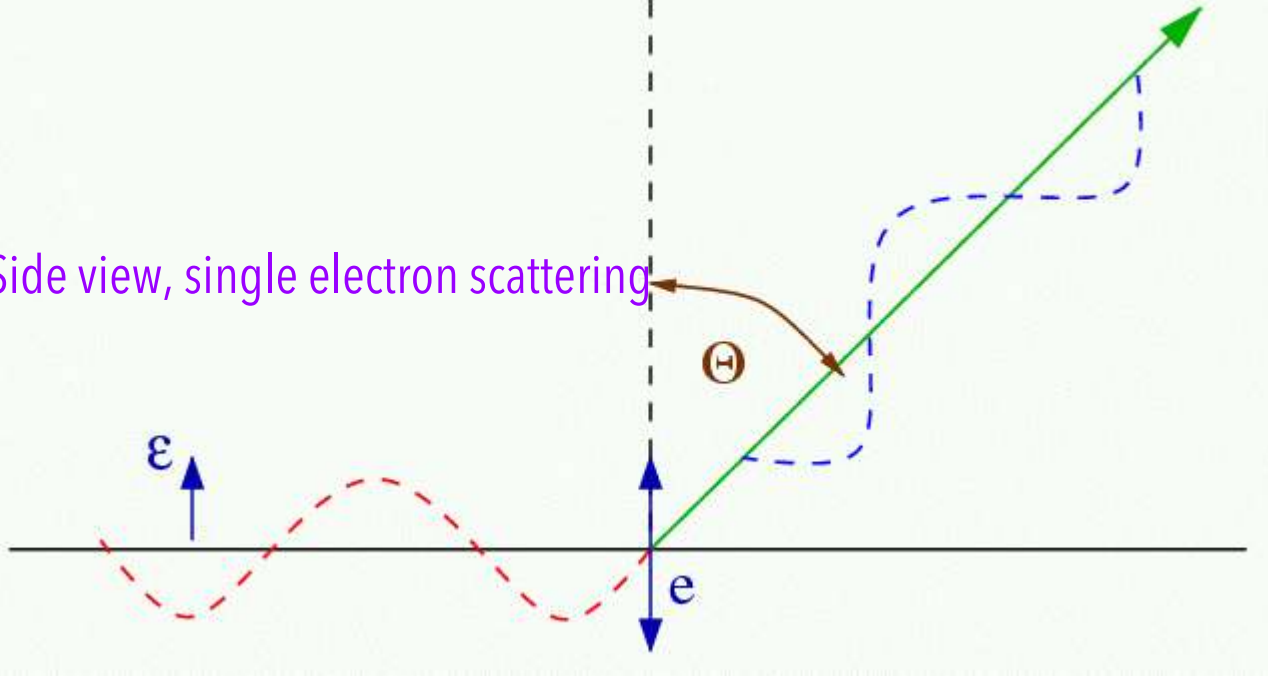
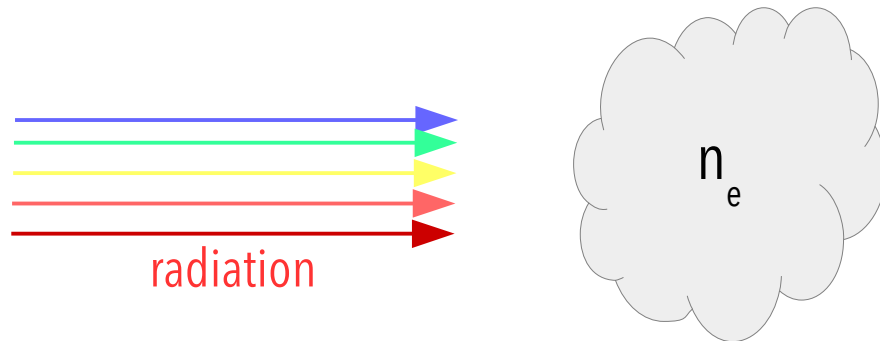
This breaks down at high energies, when classical physics needs quantum mechanics



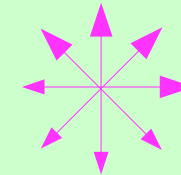
How polarization is created

<https://www.youtube.com/watch?v=fSAzPQtSAPQ> from 1:40" to 3:30"

Side view, single electron scattering



Front view: unpolarized radiation makes the electrons oscillate in all directions



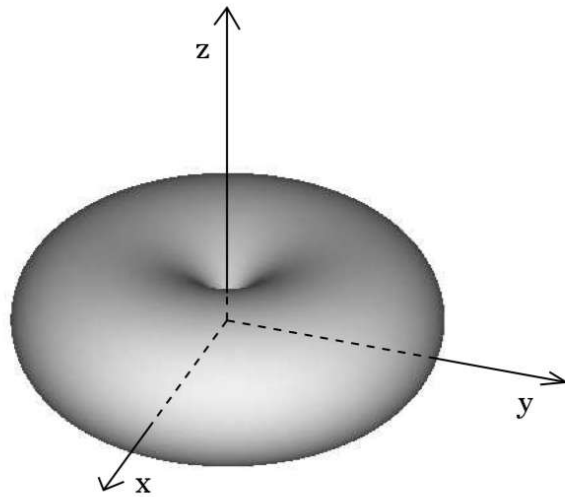
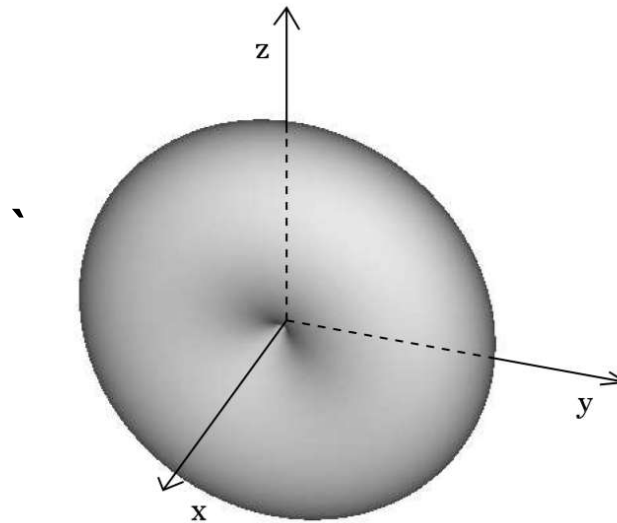
All the incoming photons follow about the same trajectory (e.g. single point



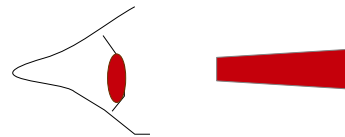
How polarization is created

The side (red) observer DO NOT receive any radiation from the "blue" oscillations

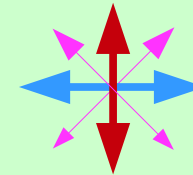
The top (blue) observer DO NOT receive any radiation from the "red" oscillations



Both observe 100% polarized radiation



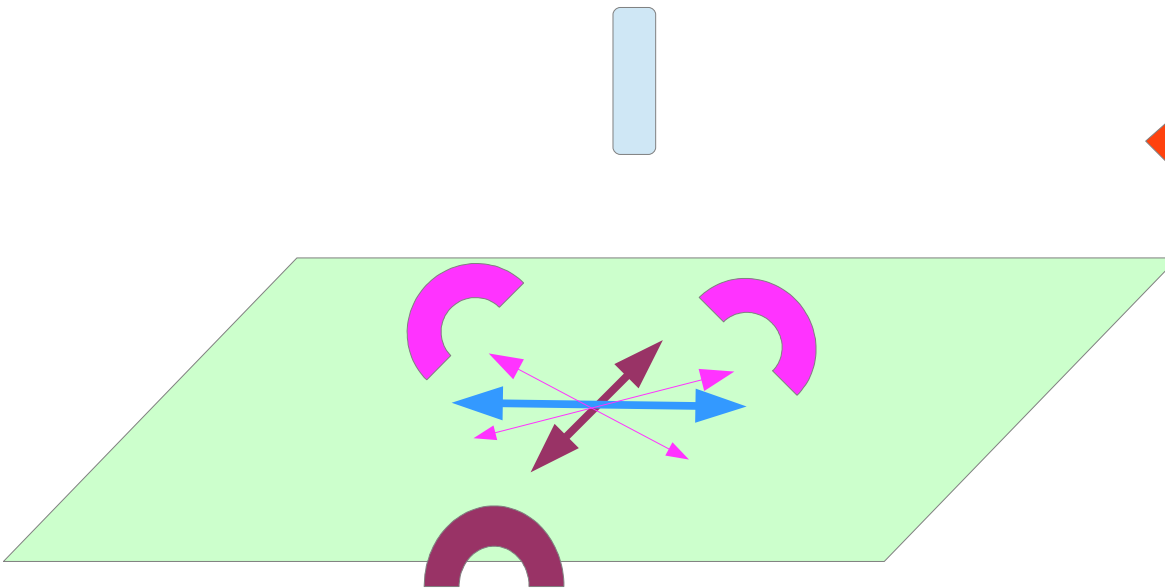
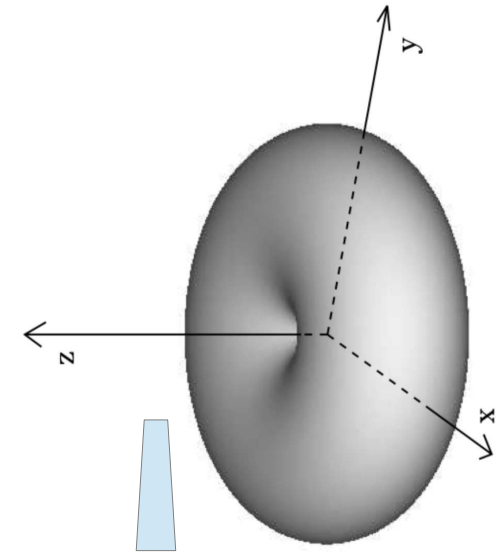
Front view: unpolarized radiation
makes the e^- 's oscillate in all directions



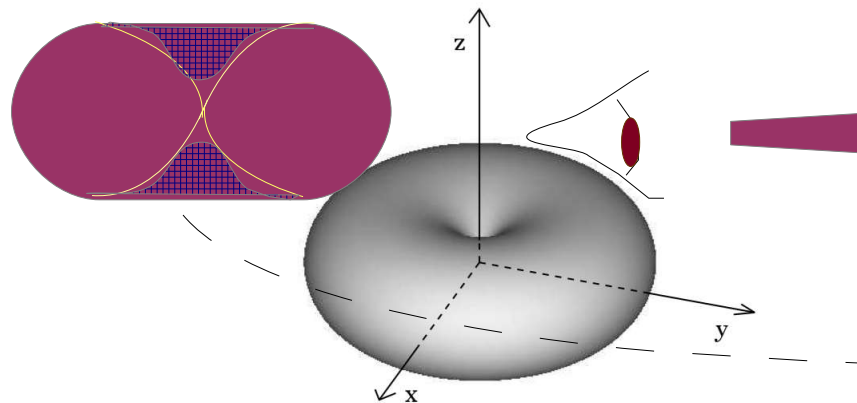


How polarization is created

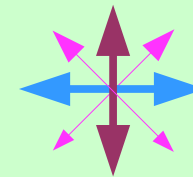
The orange observer receive partially polarized radiation



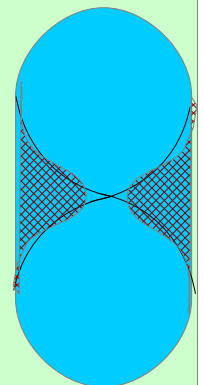
Emission pattern of the
bordaux double arrow



Front view: unpolarized radiation
makes the e^- 's oscillate in all directions



Emission pattern of the
cyan double arrow





1. Scattering symmetric wrt the scattering angle (backward=forward)
2. The amount of scattered radiation depends solely on the incident radiation

$$-\frac{dW}{dt} = \sigma_T c U_{\text{rad}}$$

3. Even if the incident radiation is unpolarized, TS can polarize scattered radiation depending on the scattering angle

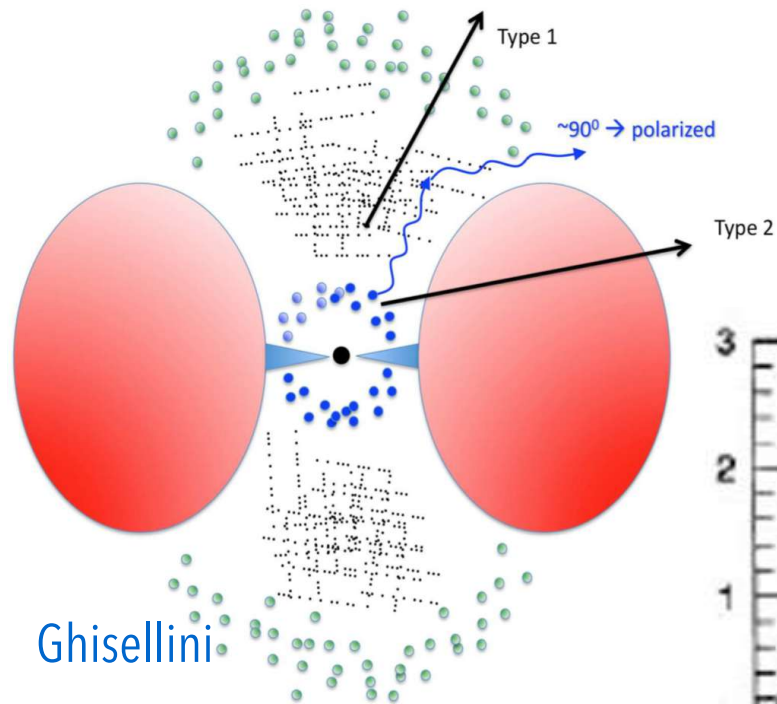
4. TS is one of the main processes capable of trapping photons in a given region.

Optical depth for TS in a medium is $\tau_T = \int \sigma_T n_e dl$

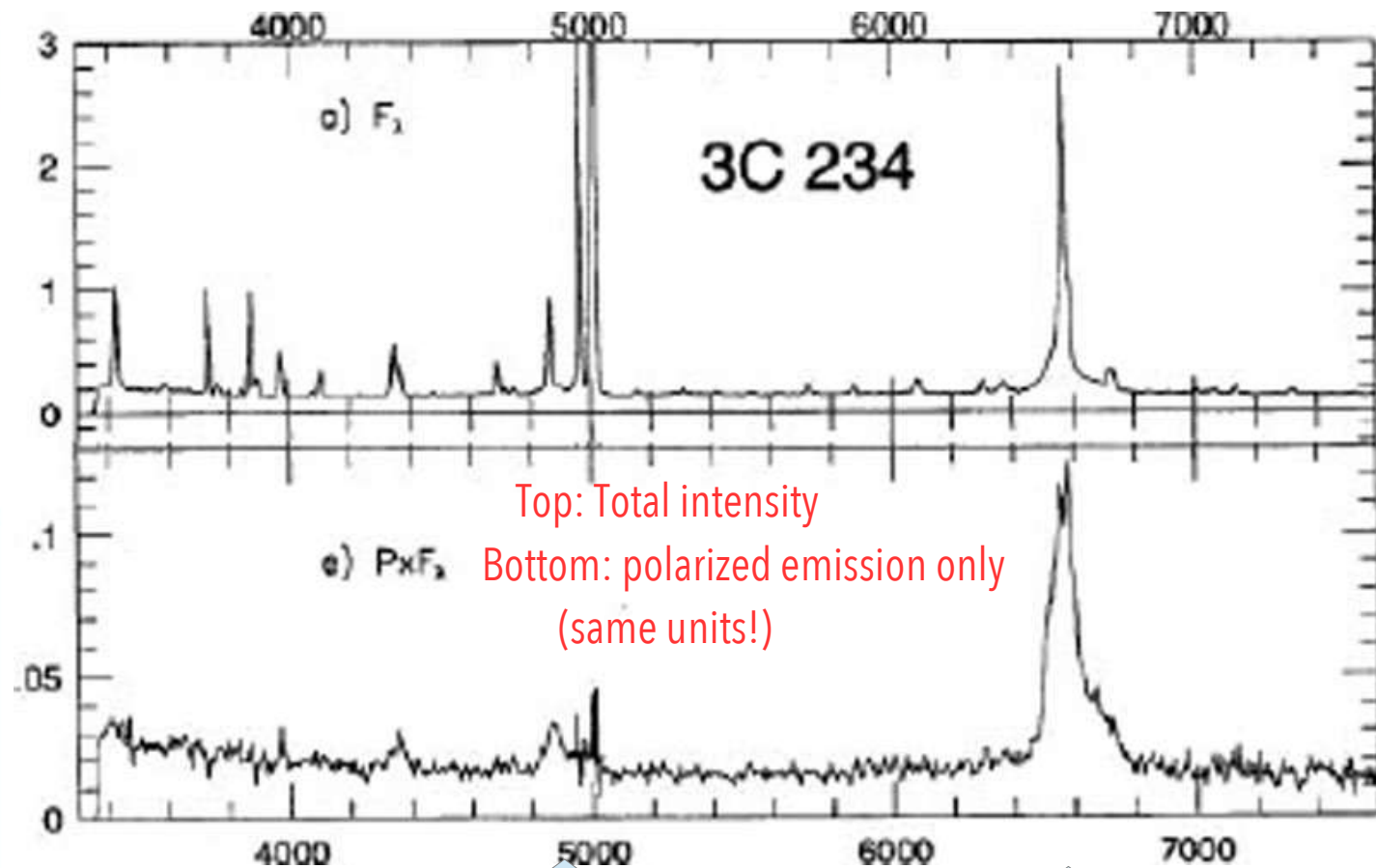
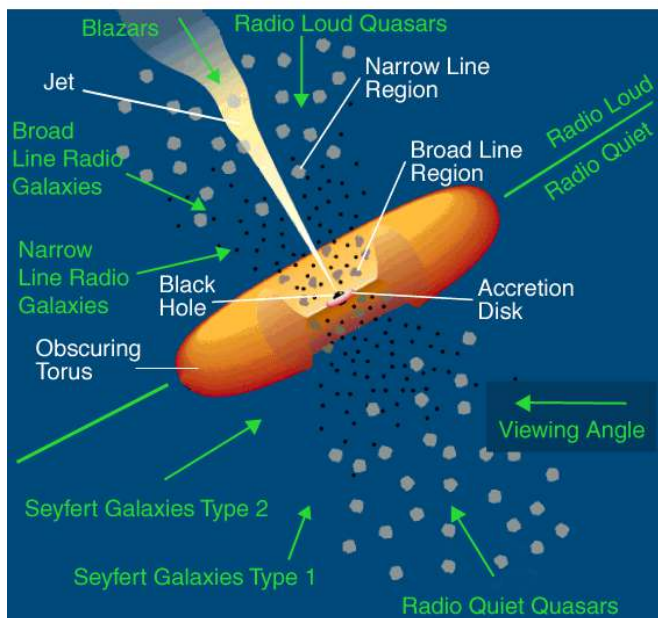
Photons are scattered in random directions, i.e. random walk in which each step is the

mean free path $\lambda_T = \frac{1}{\sigma_T n_e}$

Optical spectrum of a "type II" AGN in total intensity and in polarized light:



"broad line emission" is visible in polarized light
(when unpolarized emission is dropped)





Compton scattering

"Compton scattering or the Compton effect, is the decrease in energy (increase in λ) of a high energy (X – or γ | ray) photon, as it interacts with a free electron.

More important for astrophysics, Inverse Compton scattering also exists, where the photon gains energy (decreasing in wavelength) upon interaction with a high energy electron.

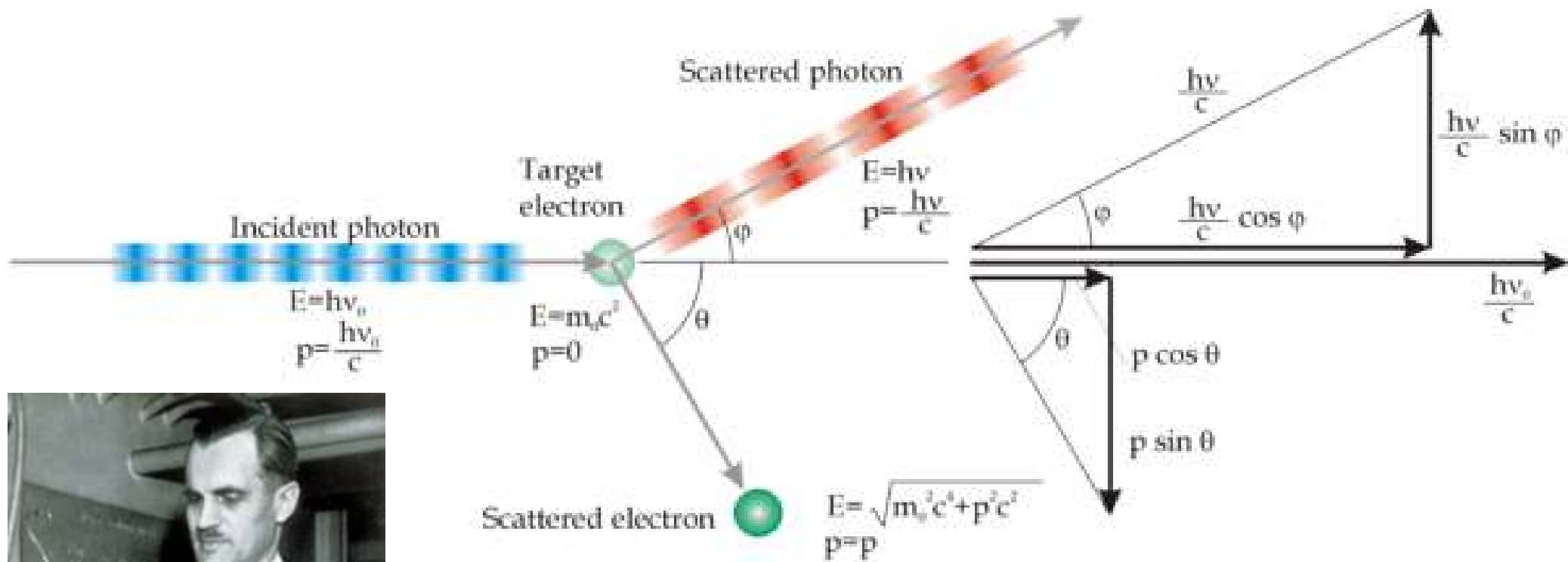
The amount the wavelength increases by is called the Compton shift.

Although nuclear Compton scattering also exists, *what is meant by Compton scattering usually is the interaction involving free electrons only.*

Compton effect was observed by Arthur Holly Compton in 1923, for which he was awarded the Nobel Prize in Physics in 1927..

The effect is important because it demonstrates that light cannot be explained purely as a wave phenomenon.

Thomson scattering, the classical theory of charged particles scattered by an electromagnetic wave, cannot explain any shift in wavelength.



A.H. Compton 1892-1962



Energy conservation:

$$h\nu_i + m_e c^2 = h\nu_f + \sqrt{p_e^2 c^2 + m_e^2 c^4}$$

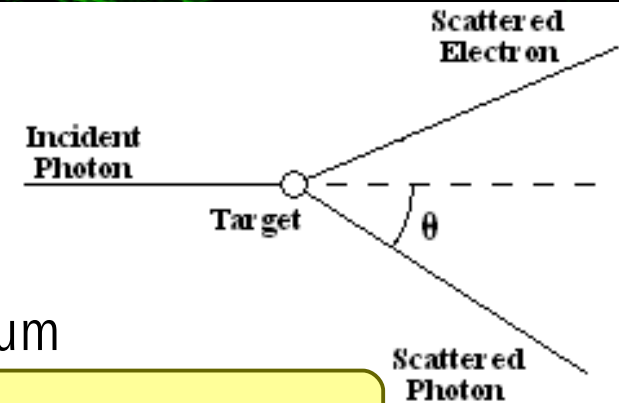
momentum conservation:

$$\vec{p}_i = \vec{p}_f + \vec{p}_e$$

$$\rightarrow p = E/c = h\nu/c \rightarrow \text{photon momentum}$$

$$\vec{p}_e = \vec{p}_i - \vec{p}_f$$

$$\rightarrow p_e^2 = (\vec{p}_i - \vec{p}_f) \cdot (\vec{p}_i - \vec{p}_f) = p_i^2 + p_f^2 - 2p_i p_f \cos \theta$$



$$p_e^2 = h^2 \nu_i^2 + h^2 \nu_f^2 - 2h^2 \nu_i \nu_f \cos \theta \quad \leftarrow \text{multiply by } c^2. \text{ Then take the square of energy conservation}$$

$$p_e^2 c^2 + m_e^2 c^4 = h^2 \nu_i^2 + h^2 \nu_f^2 + m_e^2 c^4 + 2h\nu_i m_e c^2 - 2h\nu_f m_e c^2 - 2h^2 \nu_i \nu_f$$

$$\text{Compare the two highlighting } p_e^2 c^2 \text{ we get } \rightarrow -2h^2 \nu_i \nu_f \cos \theta = 2h\nu_i m_e c^2 - 2h\nu_f m_e c^2 - 2h^2 \nu_i \nu_f$$

$$2h^2 \nu_i \nu_f (1 - \cos \theta) = 2h m_e c^2 (\nu_i - \nu_f) \rightarrow \frac{1 - \cos \theta}{m_e c^2} = \frac{\nu_i - \nu_f}{h \nu_i \nu_f} = \frac{1}{h} \left(\frac{1}{\nu_f} - \frac{1}{\nu_i} \right)$$

$$\frac{1}{\nu_f} - \frac{1}{\nu_i} = \frac{\lambda_f}{c} - \frac{\lambda_i}{c} = \frac{h}{m_e c^2} (1 - \cos \theta)$$

$$\lambda_f - \lambda_i = \frac{h}{m_e c} (1 - \cos \theta) = \lambda_0 (1 - \cos \theta)$$

Compton wavelength for electrons

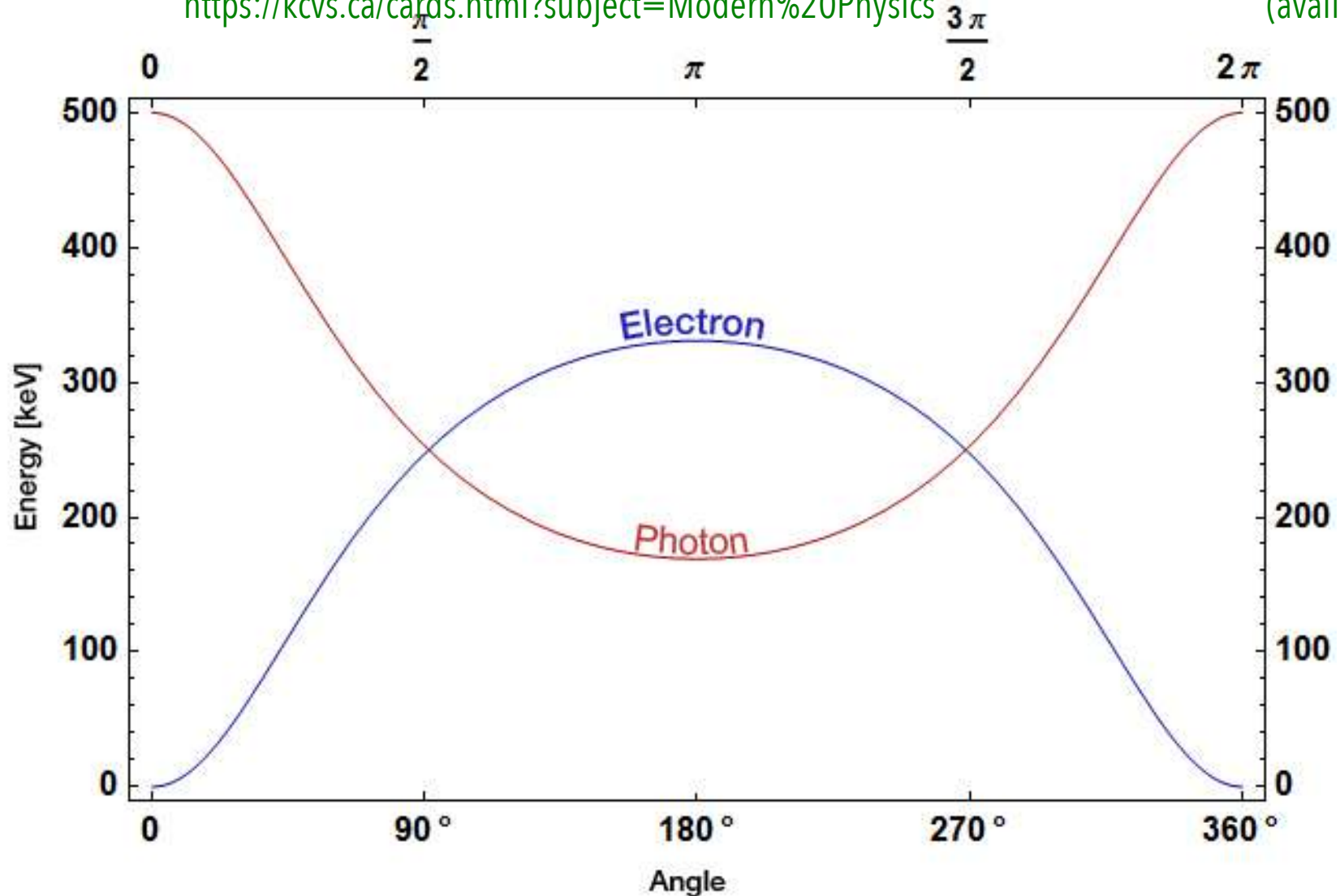
$$\lambda_0 = \frac{h}{m_e c} = 0.02426 \text{ \AA}$$

Compton scattering

a Java applet showing the geometry of the scattering can be found at

http://www.student.nada.kth.se/~f93-jhu/phys_sim/compton/Compton.htm (currently unavailable)

<https://kcv.s.ca/cards.html?subject=Modern%20Physics> (available?)





Scattered photon energy

$$h\nu_e = \frac{h\nu_i}{1 + h\nu_i(1 - \cos\theta)/m_e c^2}$$

$$\lambda_e - \lambda_i = \frac{h}{m_e c} \cdot (1 - \cos\theta) = \lambda_c (1 - \cos\theta) = 0.02426 \cdot (1 - \cos\theta) \text{ [Å]}$$

Final energy of the scattered photon is a function of:

- initial energy of the photon
- scattering angle
- the photon always loses energy, unless $\theta = 0$
- the scattering is closely elastic when $\lambda \approx \lambda_c$ (i.e. $h\nu \approx m_e c^2$)

When the photons involved in the collision have large energies, the scattering becomes less

efficient and quantum electrodynamics effects reduce the cross section.



In case $h\nu \sim m_e c^2$, the probability of interaction decreases, also as a function of the scattering angle; additional constraints come from relativistic quantum mechanics.

If we define $x \equiv \frac{h\nu}{m_e c^2}$

FYI,
Computations
never asked

$$\sigma = \sigma_T \frac{3}{4} \left[\frac{1+x}{x^3} \left(\frac{2x(1+x)}{1+2x} - \ln(1+2x) \right) + \frac{1}{2x} \ln(1+2x) - \frac{1+3x}{(1+2x)^2} \right]$$

known as the Klein - Nishina cross section:

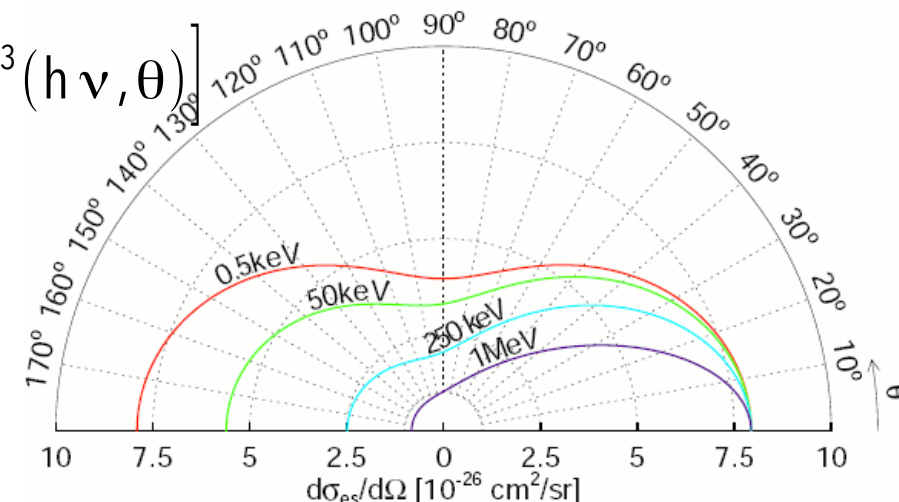
$$\text{if } x \gg 1 \rightarrow \sigma = \sigma_T \frac{3}{8} \frac{1}{x} \left(\ln 2x + \frac{1}{2} \right)$$

It originates from a differential cross section based on the scattering angle

$$\frac{d\sigma}{d\Omega} = \frac{1}{2} r_e^2 \left[P(h\nu, \theta) - P^2(h\nu, \theta) \sin^2 \theta + P^3(h\nu, \theta) \right]$$

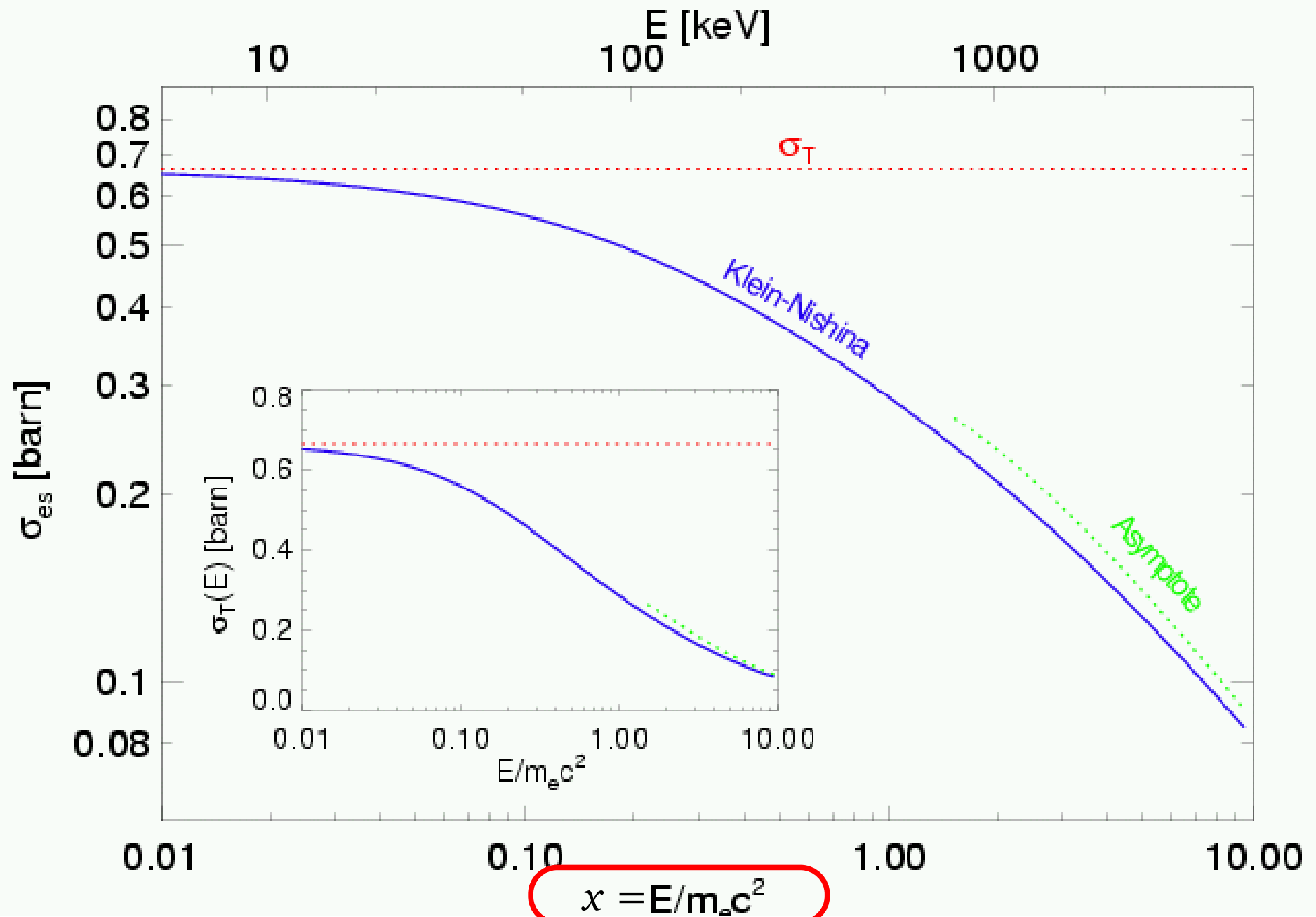
where the probability is defined as

$$P(h\nu, \theta) = \frac{1}{1 + \frac{h\nu}{m_e c^2} (1 - \cos \theta)}$$



and the Klein - Nishina cross section becomes the Thomson cross section at low photon energies

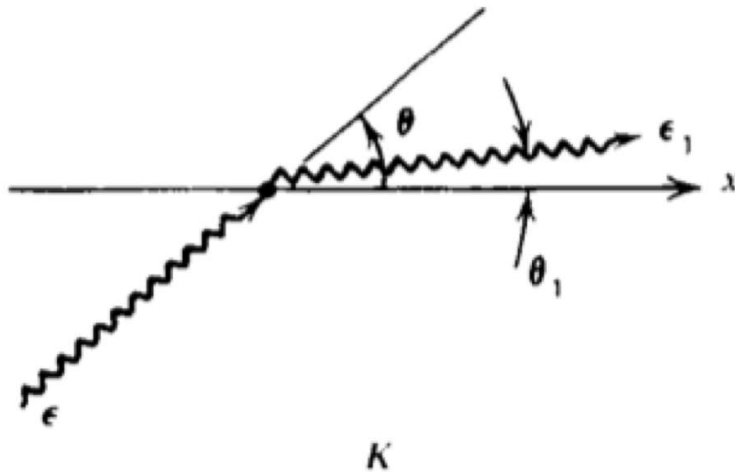
The **Klein – Nishina** cross section is relevant at high photon energies (Hard X-Rays and beyond)



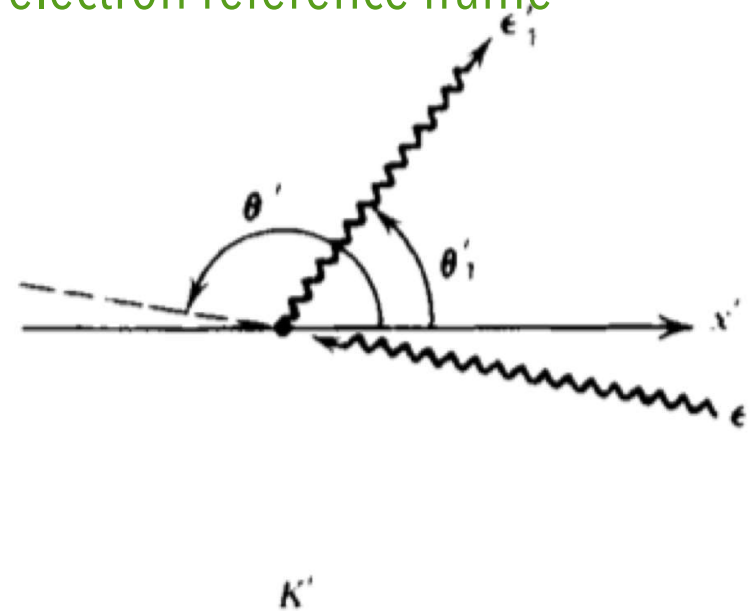
Moving, energetic (relativistic) electrons “cool down” by increasing the energy of low energy seed photons.

Remember that relativistic electrons come along their $1/\gamma$ beaming cone

Lab reference frame



electron reference frame



Moving, energetic (relativistic) electrons “cool down” by increasing the energy of low energy seed photons. To make computations simpler, it is possible to consider the centre of momentum frame where the photon energy is much less than $\sim m_e c^2$ and then $\sim \sigma_T$ can be used.

⇒ In the electron rest frame we have Thomson scattering

1. Lab system ⇒ electron rest frame:

$$h\nu_e = h\nu_L \gamma (1 - \beta \cos \theta)$$

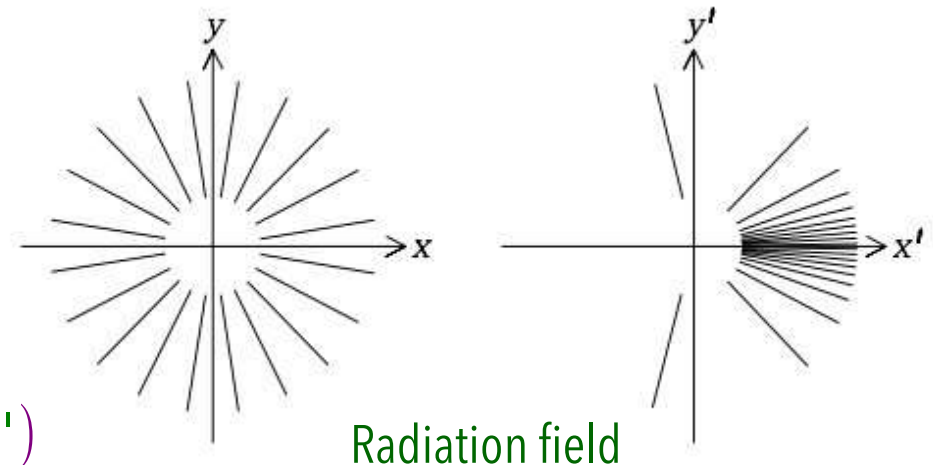
2. Thomson scattering in electron rest frame

$$h\nu_e' = h\nu_e$$

3. Electron rest frame ⇒ Lab system

$$h\nu_L' = h\nu_e' \gamma (1 + \beta \cos \theta_1')$$

$$h\nu_L' \approx \gamma^2 h\nu_L$$



i.e. ⇒ Very efficient energy transfer from electron to photon!

Angle measured in the RF of the electron



Inverse Compton scattering

The scattering angle is important for the energy of the outgoing photon:

1 — maximum energy gained by the photon if $\theta = \pi$ and $\theta_1' = 0$

☐ the photon is blue – shifted (face on collision)

2 – minimum energy for the scattered photon if $\theta = 0$ and $\theta_1' = \pi$

☐ the photon is red – shifted (end on collision)

The maximum energy that can be gained is (averaged over all angles):

$$\epsilon' \approx \frac{4}{3} \gamma^2 \epsilon$$

or, in terms of frequency

$$\nu' \approx \frac{4}{3} \gamma^2 \nu$$

In case it is possible to measure both the initial and final photon energy/frequency, then γ (or γ^2) can be deduced

$$\gamma \approx \sqrt{\frac{3}{4} \frac{\nu'}{\nu}}$$

It is a Lorentz invariant, i.e. the same in the laboratory and electron rest frame

$$E' = h\nu' = h\nu\gamma(1 + \beta\cos\theta) = E\gamma(1 + \beta\cos\theta)$$

Let's consider a region where there is a plasma of relativistic electrons and a radiation field:
in the electron framework, the photon energy is

$$\frac{d\varepsilon}{dt} = \frac{d\varepsilon'}{dt'} = c\sigma_T \frac{\langle E_{\text{rad}}'^2 \rangle}{8\pi} = c\sigma_T \gamma^2 \langle (1 + \beta\cos\theta)^2 \rangle \langle E_{\text{rad}}^2 \rangle$$

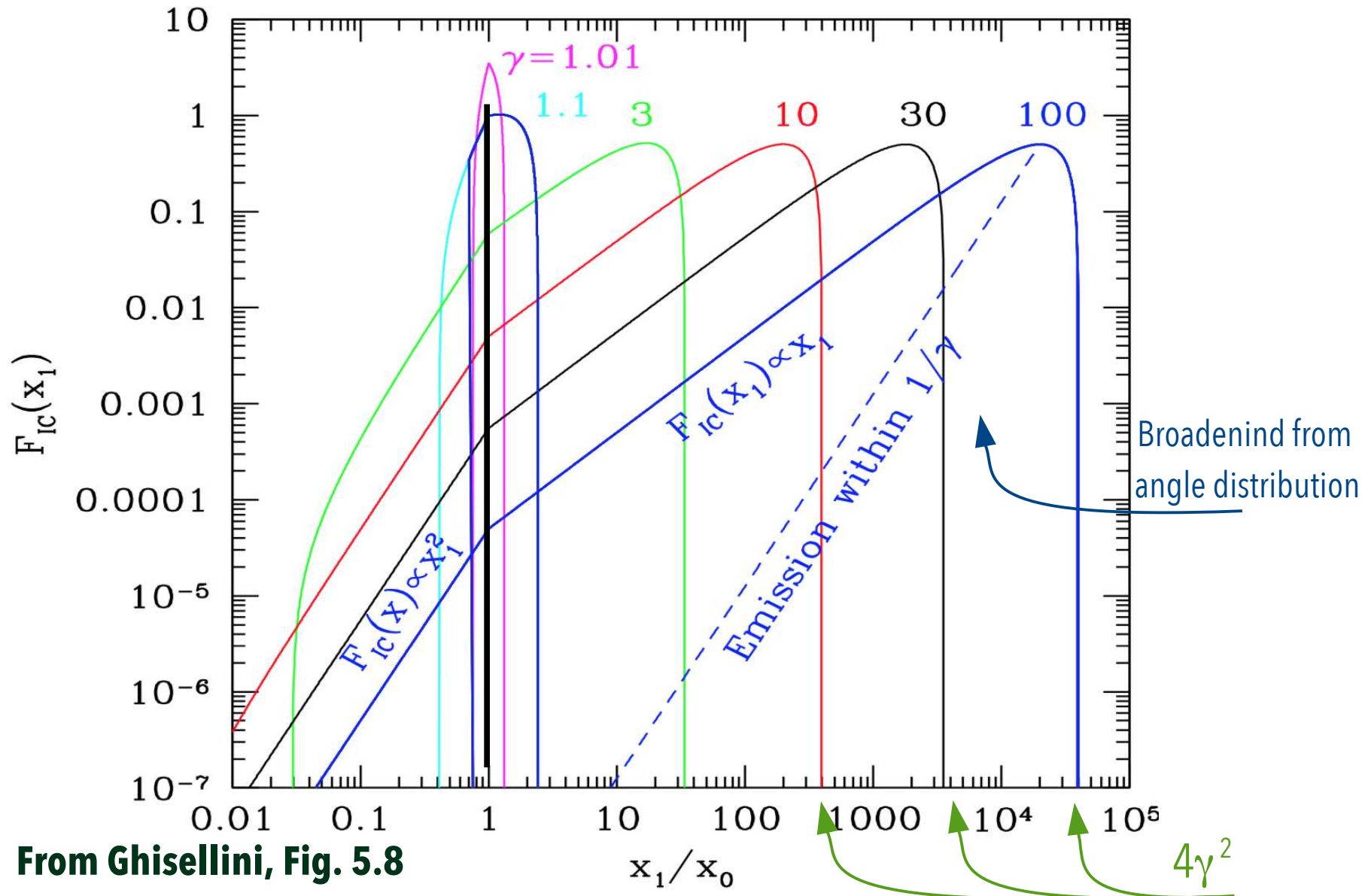
since the emitted energy is a Lorentz invariant. It is then possible to write

$$\langle E_{\text{rad}}^2 \rangle = U_{\text{ph}} \quad \langle (1 + \beta\cos\theta)^2 \rangle = 1 + \frac{1}{3}\beta^2$$

Finally, the energy coming out the scattering region is

$$\frac{d\varepsilon_{\text{out}}}{dt} = c\sigma_T U_{\text{ph}} \gamma^2 \left(1 + \frac{1}{3}\beta^2 \right)$$

Example: single energy ($x_0 = h\nu_0$) photons scattered to energies x_1 by electrons with different γ and distribution of scattering angles (N.B. All spectra are normalized, just for comparison of shift in energy)



From Ghisellini, Fig. 5.8



Inverse Compton emission .vs. Synchrotron emission

The energy associated with the seed photons, prior of the scattering is

$$\frac{d\varepsilon_{\text{in}}}{dt} = c \sigma_T U_{\text{ph}}$$

Then the net effect of the Inverse Compton scattering is

$$\frac{d\varepsilon_{\text{out}}}{dt} - \frac{d\varepsilon_{\text{in}}}{dt} = \left(\frac{d\varepsilon}{dt} \right)_{\text{ic}} = c \sigma_T U_{\text{ph}} \left[\gamma^2 \left(1 + \frac{1}{3} \beta^2 \right) - 1 \right]$$

$$\gamma^2 \left(1 + \frac{1}{3} \beta^2 \right) - 1 = \gamma^2 + \frac{1}{3} \gamma^2 \beta^2 - 1 = \gamma^2 - 1 + \frac{1}{3} \gamma^2 \beta^2 \rightarrow$$

$$\text{but } \gamma^2 - 1 = \gamma^2 \beta^2 \text{ then } \rightarrow = \gamma^2 \beta^2 + \frac{1}{3} \gamma^2 \beta^2 = \frac{4}{3} \gamma^2 \beta^2$$

FYI,
Computations
never asked

Finally we get the amount of power provided by the relativistic electrons to the seed photons

$$\left(\frac{d\varepsilon}{dt} \right)_{\text{ic}} = \frac{4}{3} c \sigma_T \gamma^2 \beta^2 U_{\text{ph}} \quad (**)$$

Let's recall the synchrotron emission $\left(\frac{d\varepsilon}{dt} \right)_{\text{syn}} = \frac{4}{3} c \sigma_T \gamma^2 \beta^2 U_H$

The same relativistic electrons radiate via synchrotron and inverse Compton

Let's compare the two losses:

$$\frac{\left(\frac{d\varepsilon}{dt} \right)_{\text{syn}}}{\left(\frac{d\varepsilon}{dt} \right)_{\text{ic}}} = \frac{U_H}{U_{\text{ph}}}$$

Their contributions add up:

$$- \left(\frac{d\varepsilon}{dt} \right)_{\text{sy+ic}} = C_s \varepsilon^2 \left[H^2 + 8\pi U_{\text{rad}} \right]$$

The increased cooling rate implies that the **electron radiative lifetime is consequently reduced**

$$t_{\text{sy+ic}}^* \approx \frac{1}{\varepsilon^*} \frac{1}{H^2 + 8\pi U_{\text{rad}}} \approx \frac{3 \cdot 10^8}{(H^2/8\pi + U_{\text{rad}}) [\text{eV cm}^{-3}]} \frac{1}{\varepsilon^* [\text{GeV}]} \text{ yr}$$

Therefore.... hard times for relativistic electrons!

(Wait a little while for an example: radiative lifetime for a relativistic electrons in a radio lobe.)

$H_{\text{eq}}, U_{\text{rad}}$, have strong dependence on z !, i.e. **when**

They have strong dependence on the location in the Universe, i.e. **where**

How to evaluate U_{rad} ?

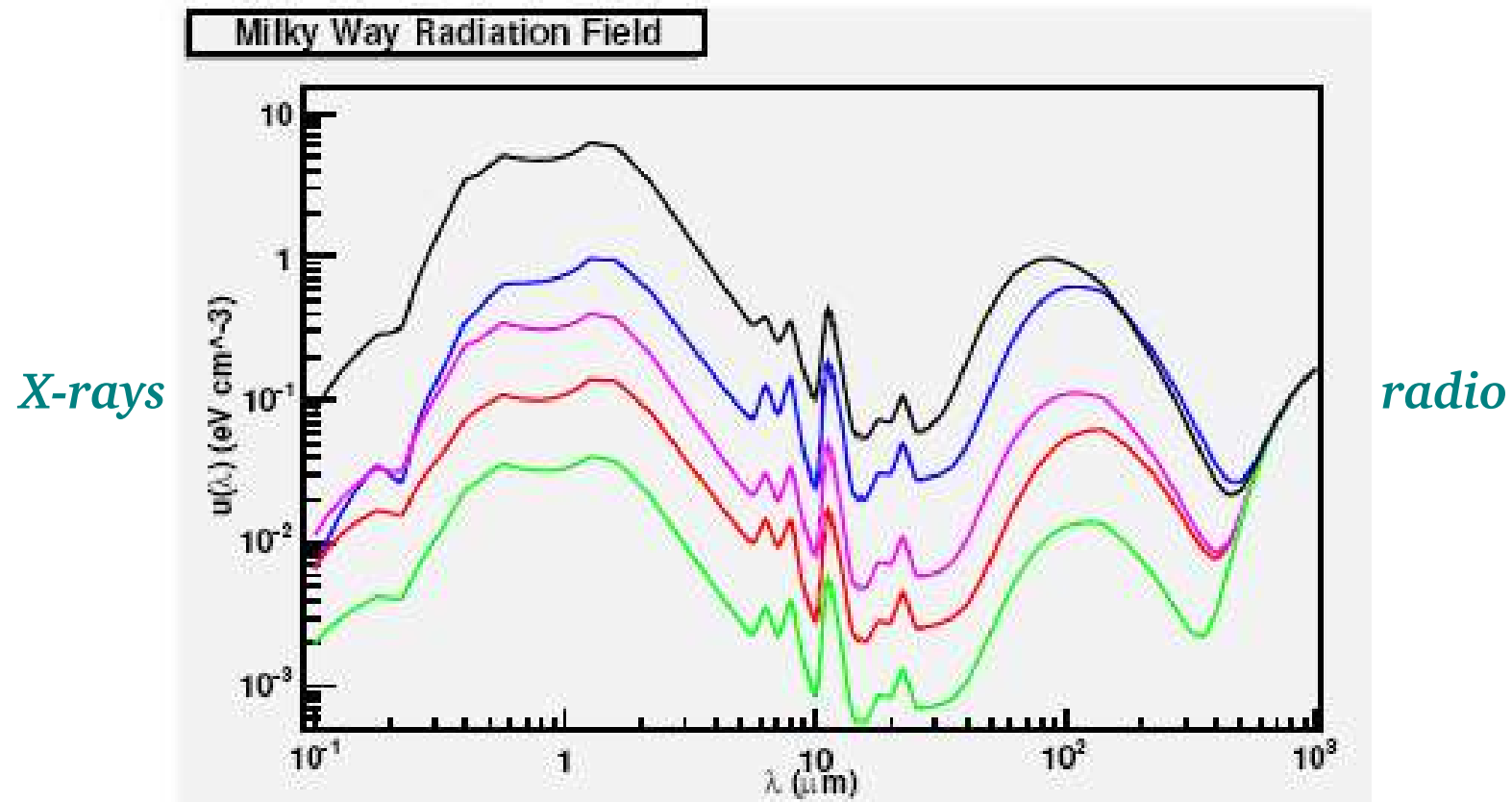
CMB is a radiation which can be found in every corner of the Universe

$$H_{\text{CMB}} \sim 3.25(1+z)^2 \mu\text{G}$$

H_{CMB} is the equivalent to the H field strength necessary to have the same losses for Synchrotron and IC with CMB photons

The "Local Radiation Field"

Evaluation of U_{rad} : as taken in a number of selected locations in/around the Milky Way.



Spatial variation of the total radiation field throughout the Galaxy.

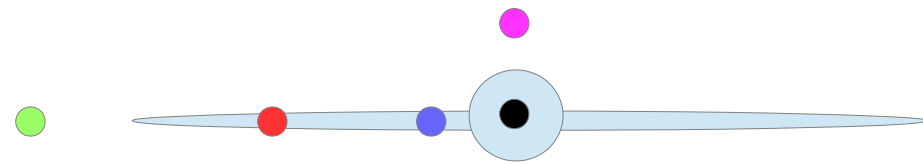
Black line : total radiation field for GC.

Magenta line : total radiation field for $R=0$, $z=5$ kpc,

Blue line : total radiation field for $R=4$, $z=0$ kpc,

Red line : total radiation field for $R=12$, $z=0$ kpc,

Green line : total radiation field for $R=20$, $z=0$ kpc.



This diagram will be discussed a bit further when considering the

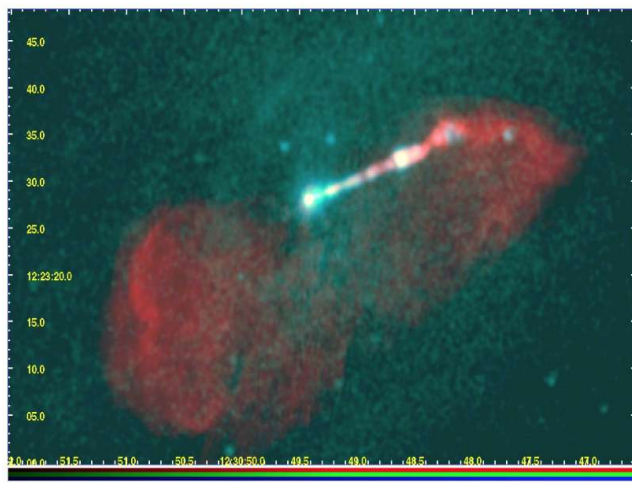
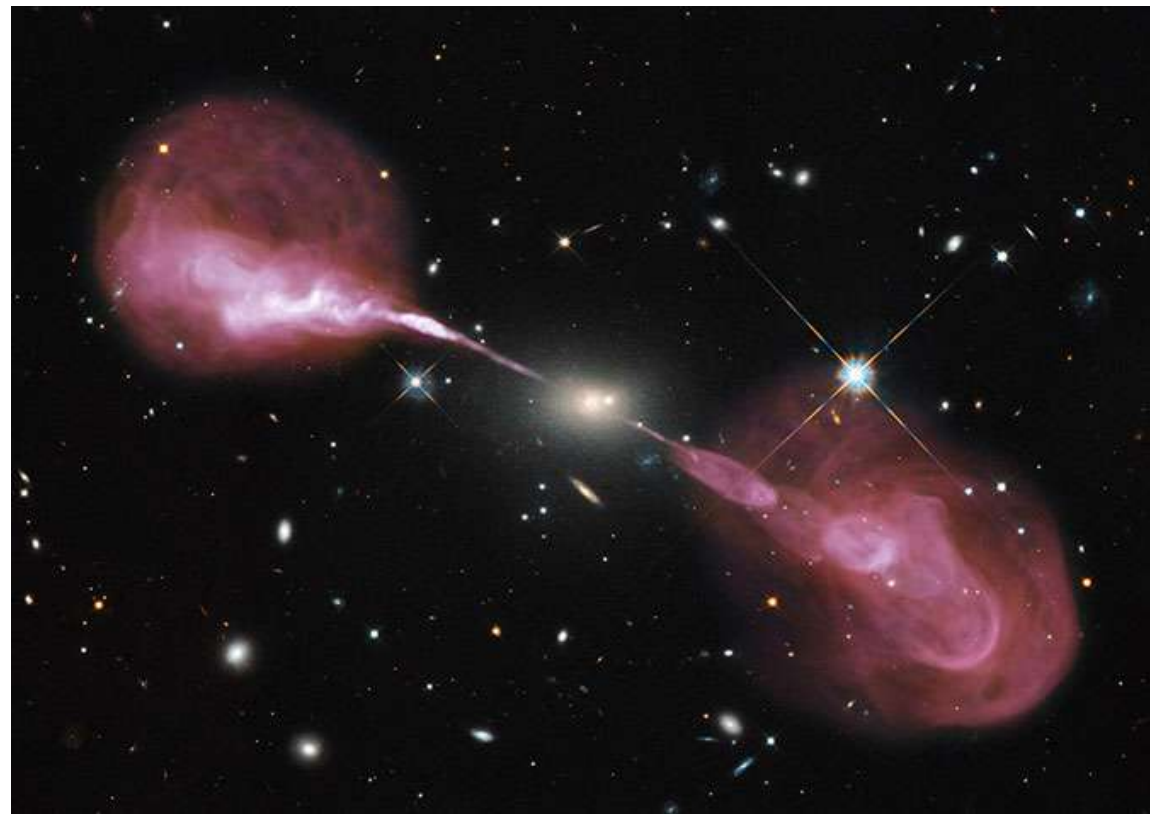
If U_{rad} can be determined, then the magnetic field can be measured!

If $U_{\text{rad}} = U_{\text{CMB}}$ as taken in a number of selected locations given the typical γ of the electrons, in general, the ratio $L_{\text{radio}}/L_{\text{X}}$ is a measure of H^2/U_{CMB}^2 in the radio emitting region.

$$\frac{L_{\text{radio}}}{L_{\text{X}}} \simeq \frac{U_{\text{H}}}{U_{\text{rad}}}$$

N.B. The energy density of the synchrotron radiation produced by the magnetized plasma is generally much lower than U_{CMB} .

The lobes of (giant) radio sources are suitable sites for this kind of studies





Line transitions

1. *Electron changes its binding energy in an atom (bound-bound transition)*
 - 1b. *Electron changes its binding energy in a molecule (bound-bound transition)*

2. *A molecule changes its binding energy (vibrations)*

3. *A molecule changes its rotational energy*

→ **ALL PROCESSES HAPPENS VIA EMISSION/ABSORPTION OF RADIATION**

(but it is not the whole story...)

Quantum mechanics is at work:

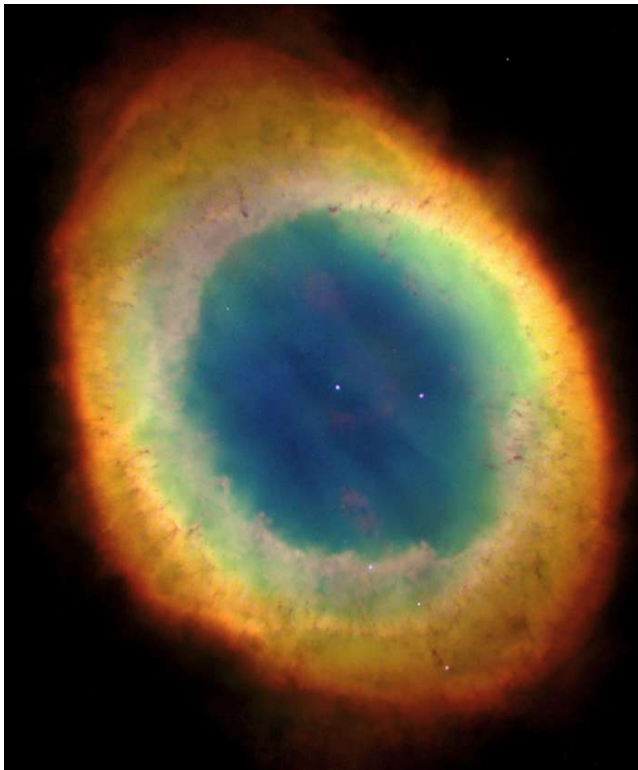
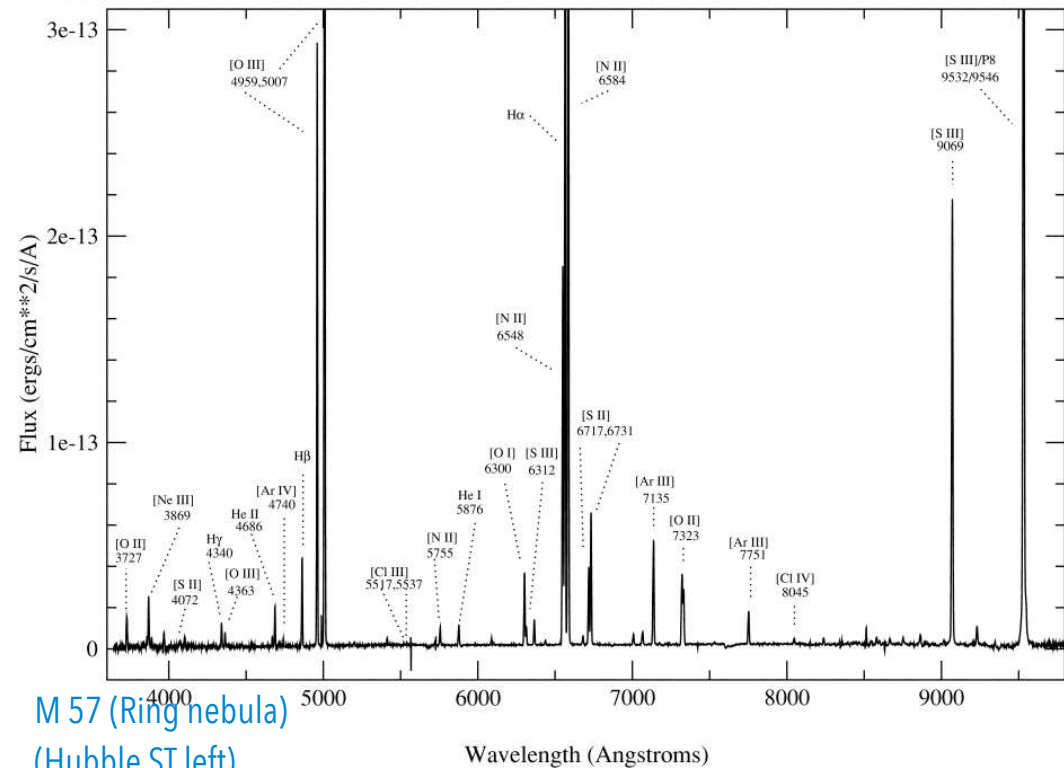
*Discrete energies are relevant $\Rightarrow$ radiative transitions produce line emission/absorption
no continuum (except ionization/recombination)*

Statistical laws in case of Thermal Equilibrium (TE, at least Local LTE)

Textbooks and further reading

1. Dopita & Sutherland: chap. 2 (more arguments in other chapters)
2. Rybicki – Lightman: chap. 1.6 (Einstein's Coefficients); chap 9; chap10; chap 11
3. Tennyson: (H spectrum & similarities, some molecular line theory/observations)

<https://www.unitronitalia.com/prodotti/analysis-and-interpretation-of-astronomical-sp.pdf>



M 57 (Ring nebula)
(Hubble ST left)



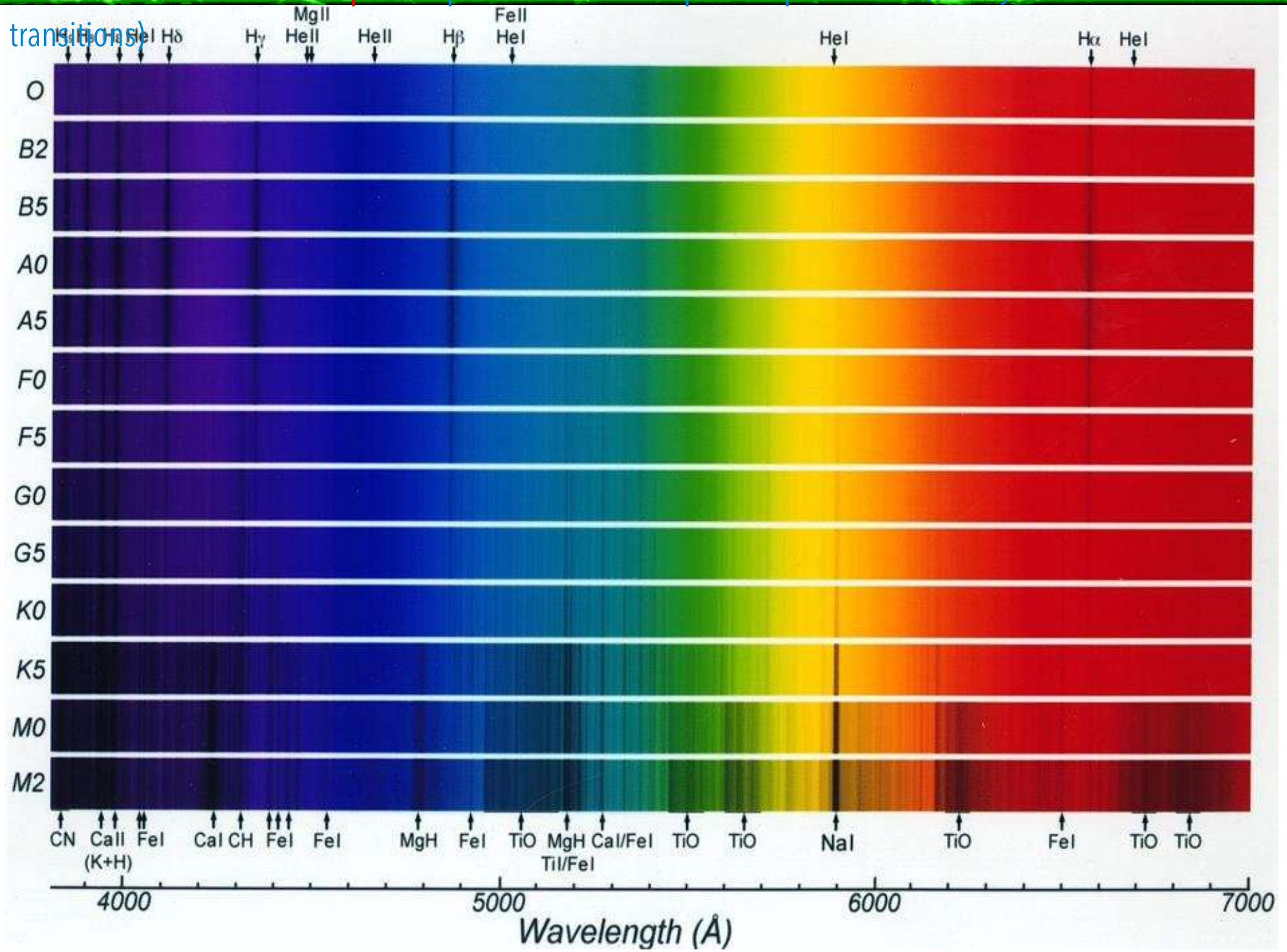
What can we learn?

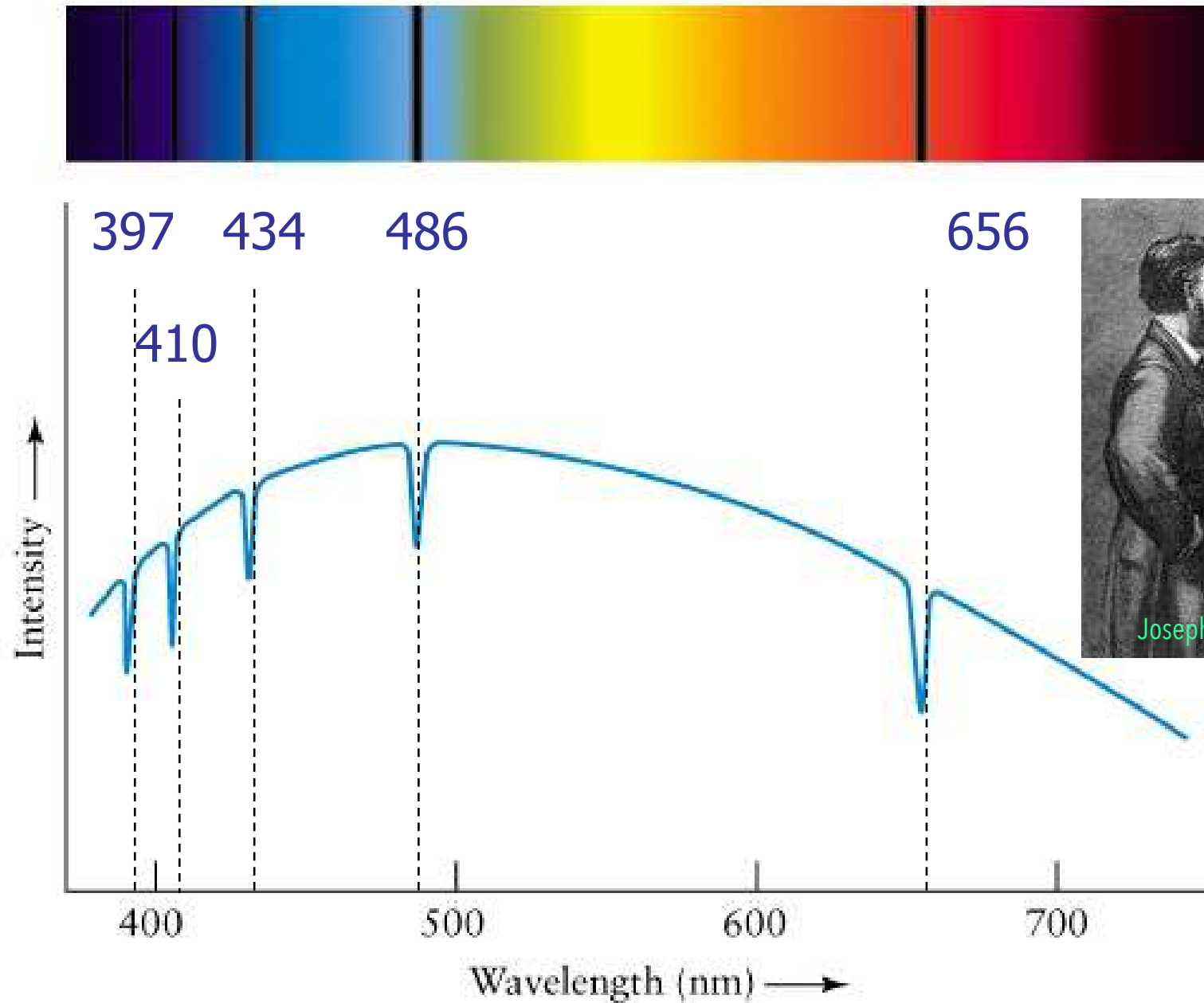
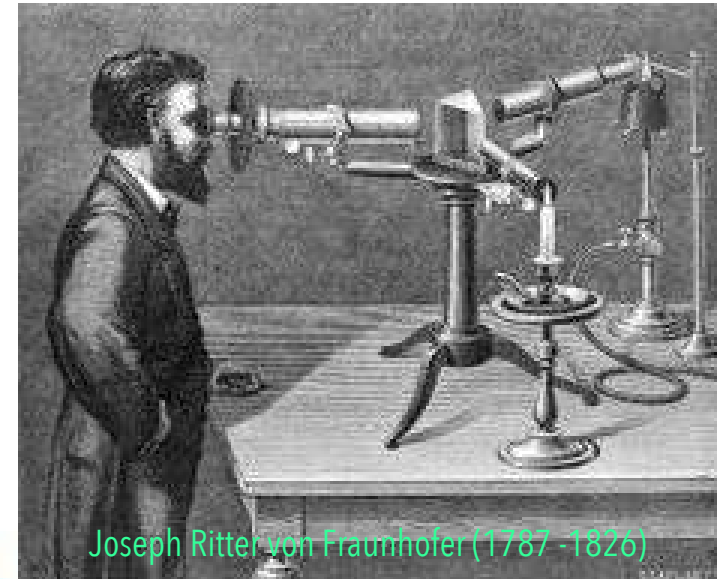
- 1. Composition of a celestial body*
- 2. Temperature (and other gas/plasma properties)*
- 3. Abundances of various species (intensity of transitions) and excitation/ionization state*
- 4. Motions:*
 - 4.1 particles within a cloud (line widths)*
 - 4.2 clouds within a celestial body (line shift)*
- 5. Pressure (relation with collisions, line broadening, turbulence)*
- 6. Magnetic field (degenerate state, angular momentum becomes split if H is present)*



Outline

0. *Examples of lines in spectra of celestial bodies*
1. *Very basic definition of tools (atomic structure, orbits/als, energy levels, transitions,...)*
2. *Einstein's Coefficients (A_{ul} , B_{ul} , B_{lu})*
3. *Spectral lines: width, centroid, shift*
4. *Permitted, semi-permitted/forbidden, forbidden transitions*
5. *When density is a key player, and competing transitions!*
6. *Detailed statistical equilibrium and (the clash with) thermal equilibrium*
7. *Neutral hydrogen: a case study*

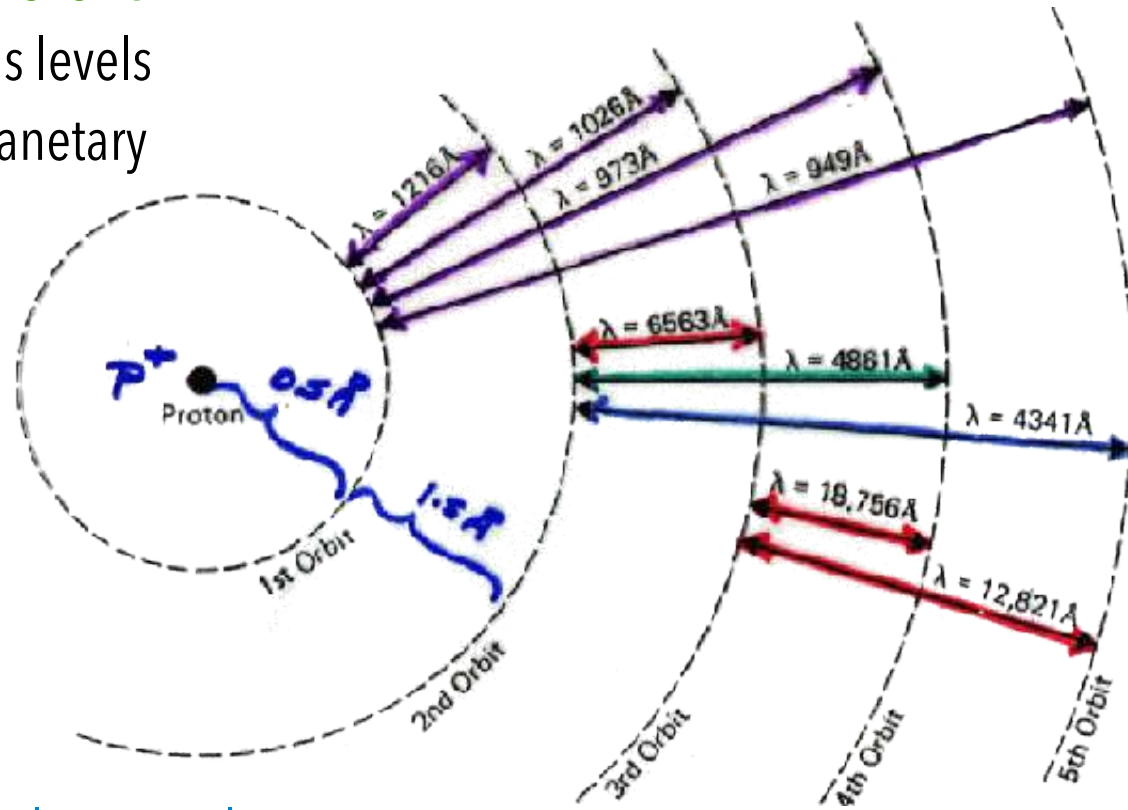




Each electron **bound** to its nucleus: **binding energy** is **quantized** into various **LEVELS** (as predicted by QM), corresponding to **different distances to the nucleus**, i.e. the various levels correspond to different orbits in a sort of planetary system.

The **outermost electron may** change its binding energy, swapping between two LEVELS, either acquiring (absorption) or releasing (emission) energy **$h\nu$** via **radiative transitions**

Such radiative transitions must obey strict **selection rules**.



(warning... there are also **other types of energy transitions!**)
(They will be considered later on)



Basic concept of atomic structure (2)

The **Ground State** corresponds to the maximum binding energy (which is defined as negative).

Electrons in states other than the Ground State are defined "**Excited**"

To **IONIZE** an atom in the ground state an energy in excess to this maximum binding energy must be provided.

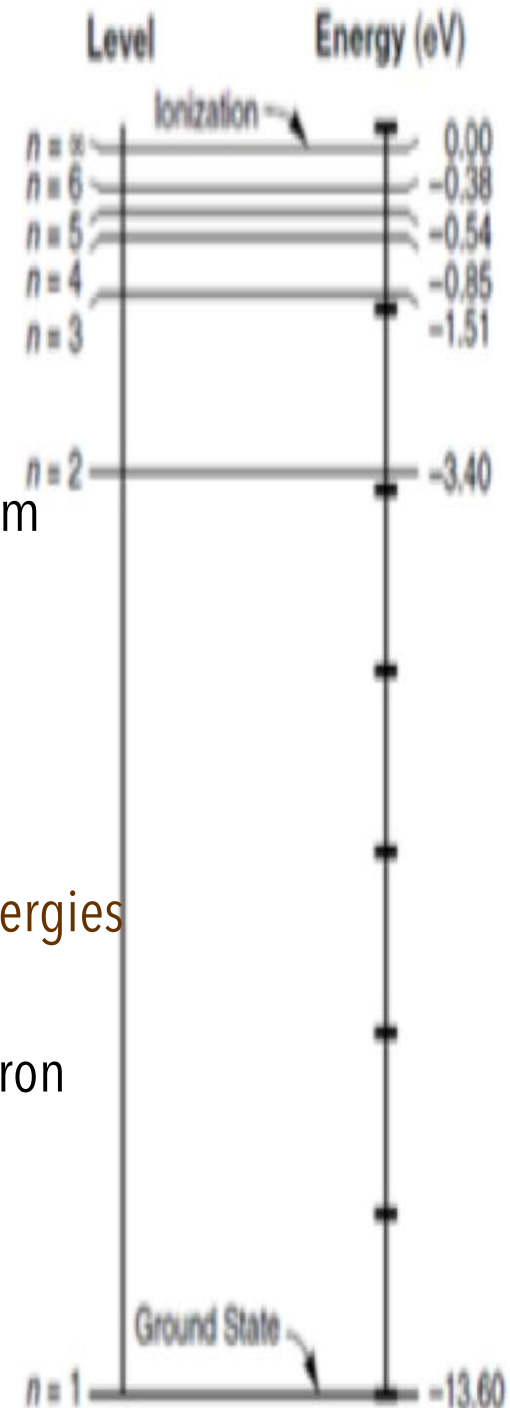
The maximum binding energy defines the minimum ionization energy.

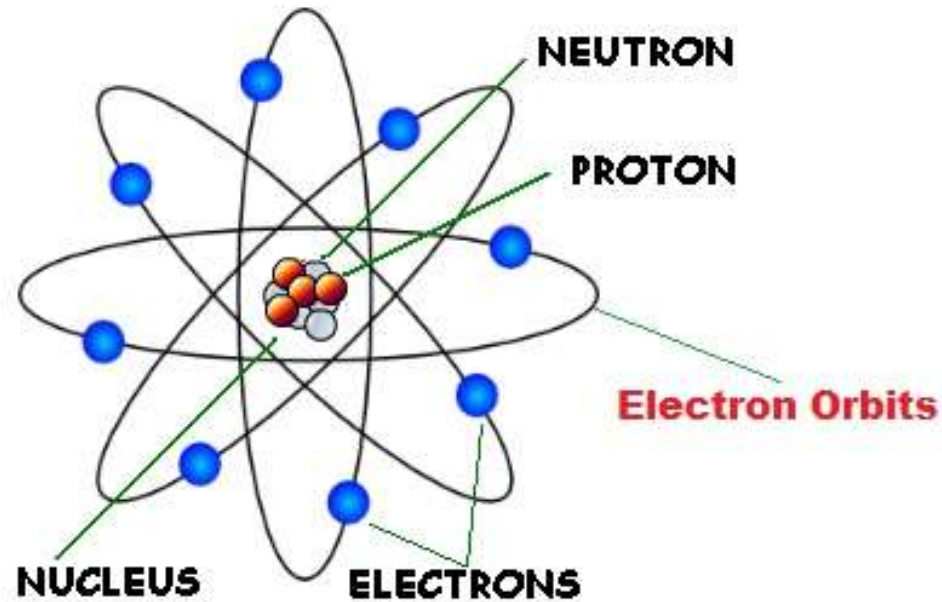
E.g. for an H atom it is 13.6 eV (UV photons, 912 Å)

Electrons in states different from the ground state have lower ionization energies

[In general] in polyelectronic atoms, only the outermost (less bound) electron is considered

Obviously, real atoms are more complicated!





(artist's view of) Bohr's model for O

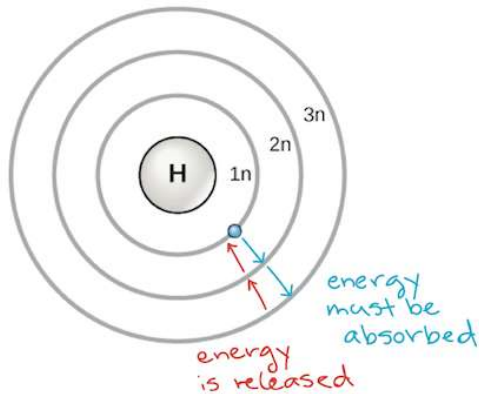
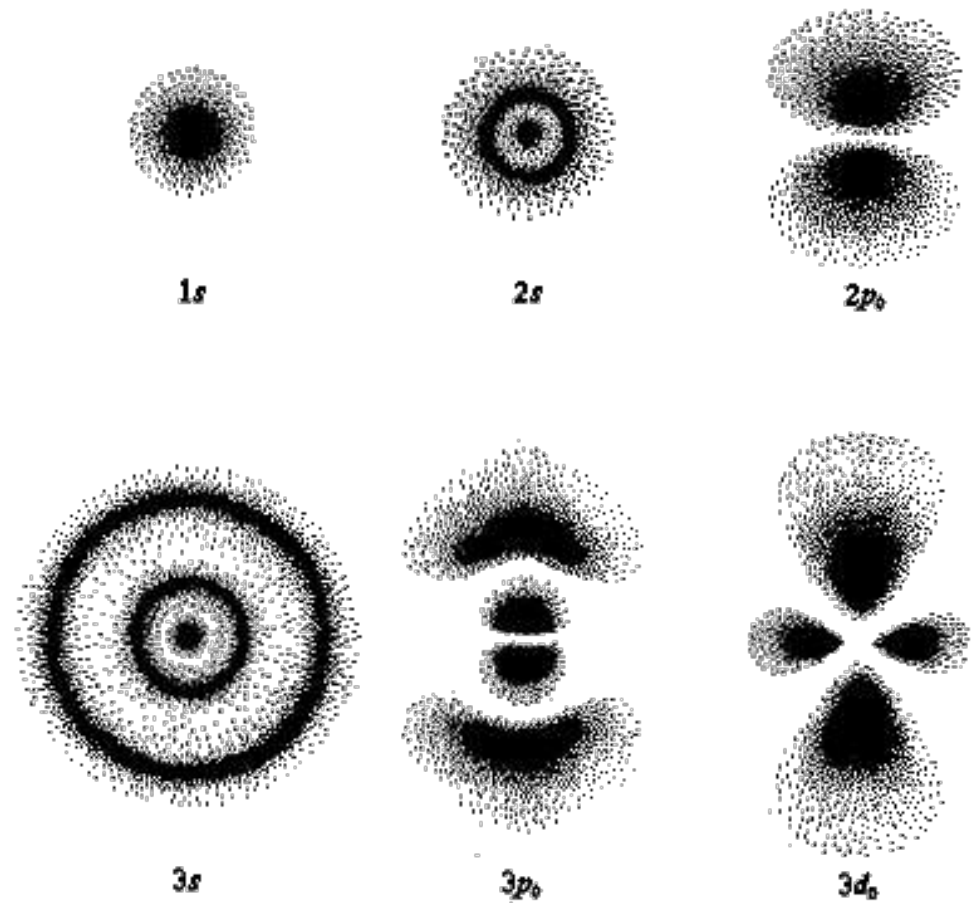


Figure 6-12. Probability density plots of some hydrogen atomic orbitals. The density of the dots represents the probability of finding the electron in that region.

© 1983 University Science Books; "Quantum Chemistry" by Donald A. McQuarrie



Quantum numbers:

n - main number = 1,2,3,... defines the **energy** and the **size** of the "orbit"

$$a_n = a_0 \frac{n^2}{Z}$$

$$a_0 = 0.53 \text{ \AA}$$

l - azimuthal number (angular momentum, related to eccentricity, defines the **SHAPE**)

0,1,2,...,n-1

$$\sqrt{1-e^2} = \frac{b}{a} = \frac{l+1}{n} \leq 1$$

m - magnetic number (orbit orientation, relevant in case of magnetic field)

-l, -l+1, ..., 0, ..., l-1, l

➡ However... at a given main number (n), there is a tiny difference in energy between levels with different combination of l,m

➡ this is not true (first order) for hydrogen, where there is a high degree of energy degeneracy (fine structure)

further extremely small energy structures (**hyperfine structure**) can be defined by the

spin quantum number $s = (\pm 1/2)$

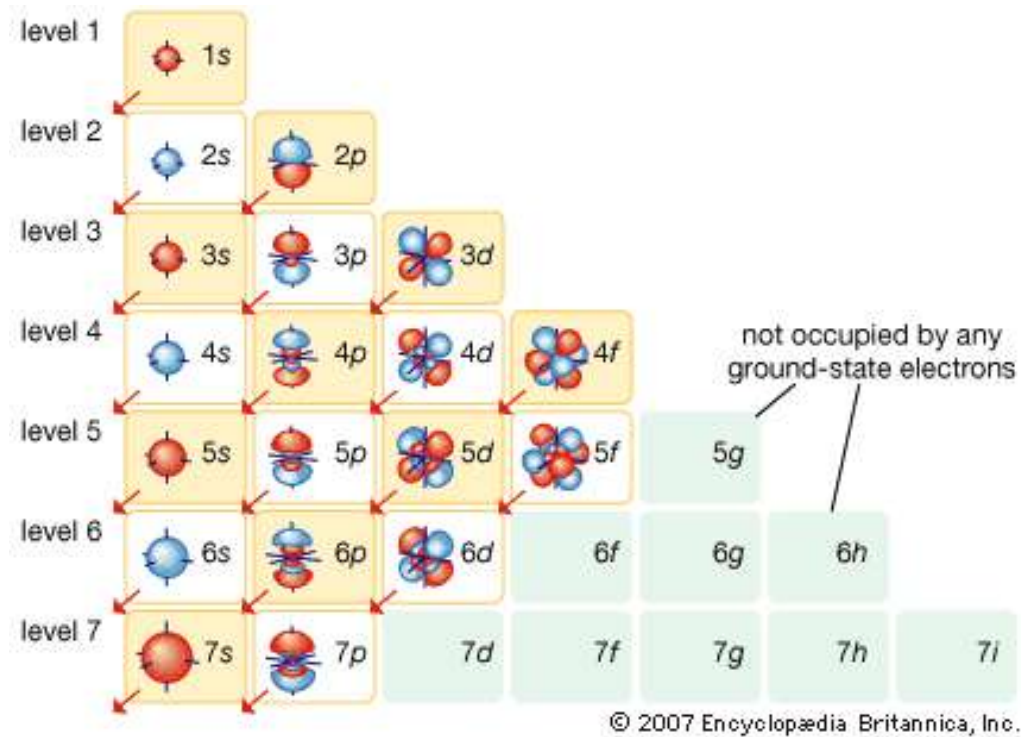
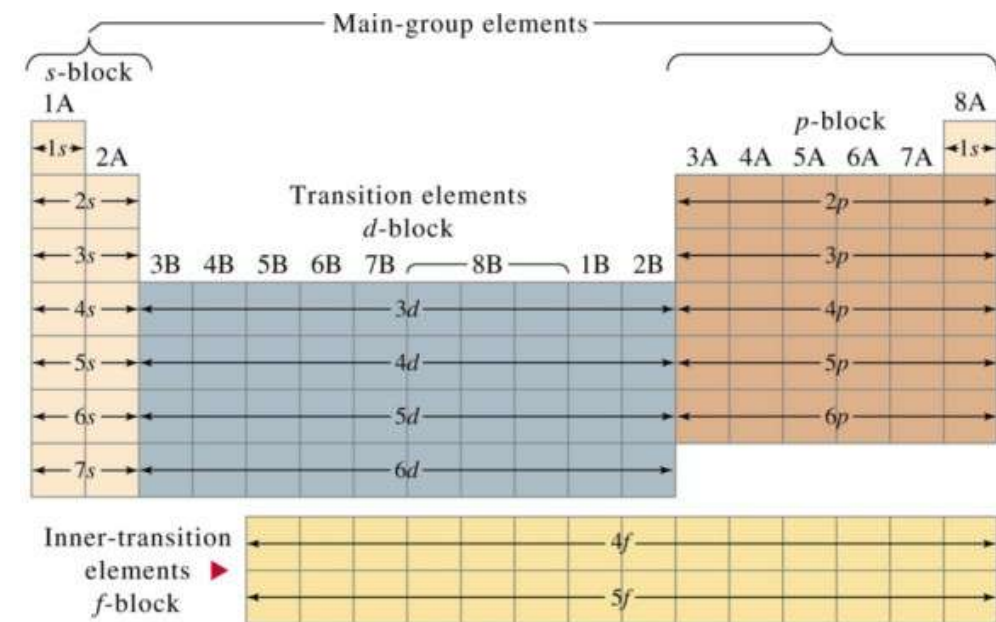
contrary to classical theory of charges in motion, the electrons do not radiate in their curved orbits, except during a "transition"



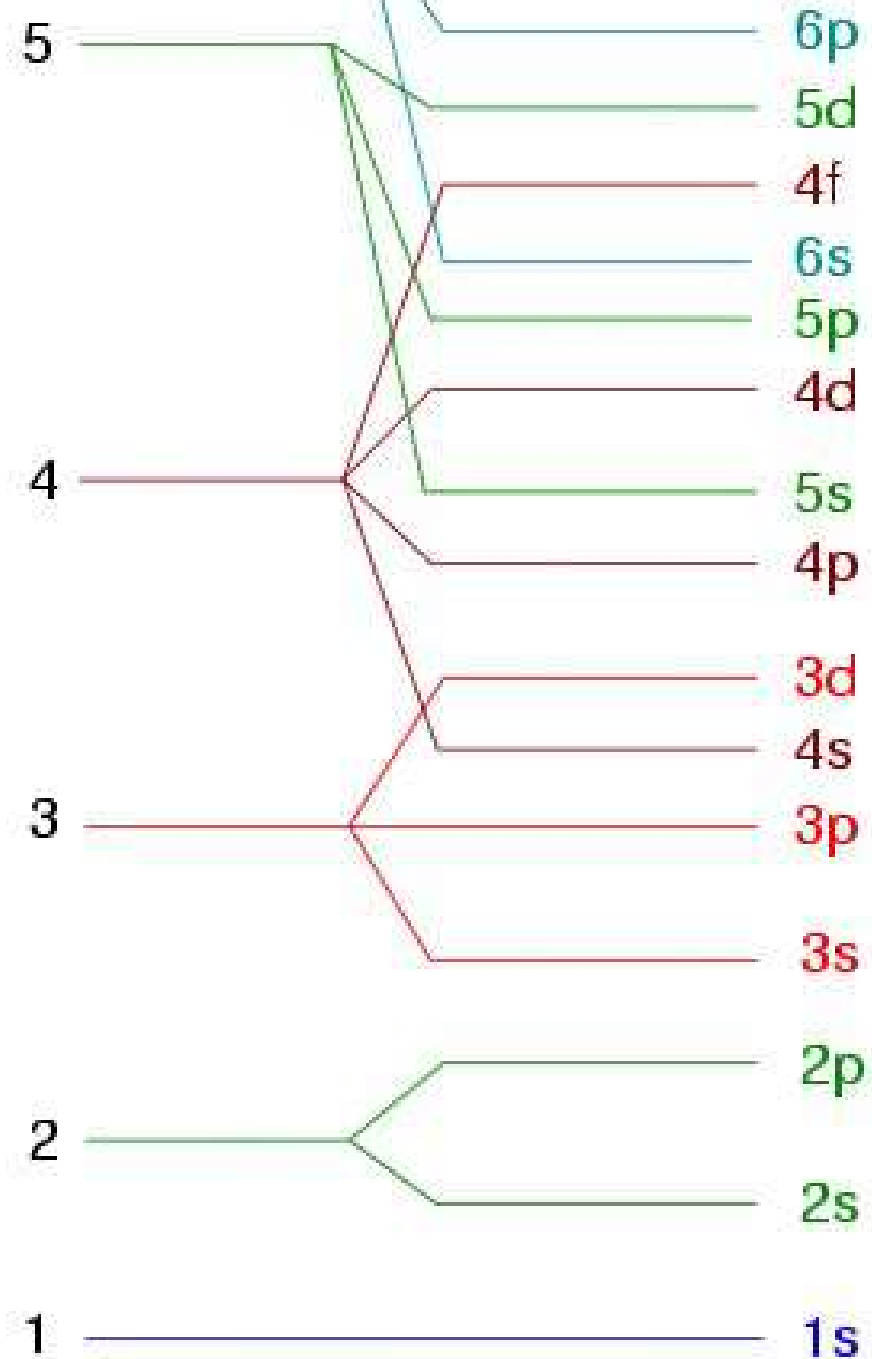
	s ($\ell = 0$)	p ($\ell = 1$)			d ($\ell = 2$)					f ($\ell = 3$)						
	m = 0	m = 0	m = ±1		m = 0	m = ±1		m = ±2		m = 0	m = ±1		m = ±2		m = ±3	
	s	p _z	p _x	p _y	d _{z²}	d _{xz}	d _{yz}	d _{xy}	d _{x²-y²}	f _{z³}	f _{xz²}	f _{yz²}	f _{xyz}	f _{z(x²-y²)}	f _{x(x²-3y²)}	f _{y(3x²-y²)}
n = 1	.															
n = 2	.															
n = 3	.															
n = 4																
n = 5										...						
n = 6					...					...						
n = 7		...			...					...						



Basic concept of atomic structure (6): Filling the orbitals



Energy



Hydrogen:

simplest example for studying lines between various energy levels

The frequency ν_{UL} of a photon emitted/absorbed during a transition

between two levels U and L is

$$\nu_{UL} = c R_H \left(\frac{1}{L^2} - \frac{1}{U^2} \right) \text{ Hz} ; \quad U > L > 0$$

$$R_H = \frac{2\pi^2 e^4 m_e}{h^3 c} = 1.1 \cdot 10^5 \text{ cm}^{-1} \rightarrow \text{Rydberg constant for H}$$

② increasing U , lines get closer to a limiting frequency $\nu = \frac{c R_H}{L^2}$

② if L is large, lines of different series start at very close frequencies and produce a spectrum similar to a continuum (but it is not!)

[hydrogen-like atoms, He^+ , Li^{++} , Be^{+++} , ... obtained assigning the energy E_n at each "atom"]

$$E_n = -Z^2 \frac{R_H h c}{n^2} + \epsilon(n, l)$$



Hydrogen transitions in terms of energy:

$$R_y = \frac{2\pi^2 e^4 m_e}{h^2} = 13.6 \text{ eV} \quad (\text{called 1 Rydberg (energy unit)} = 2.18 \cdot 10^{-18} \text{ J})$$

$$h \nu_{UL} = R_y \left(\frac{1}{L^2} - \frac{1}{U^2} \right) \text{ eV} \quad ; \quad U > L > 0$$

For a given pair of quantum numbers, decay and excitation can either generate **(emission)** or cancel **(absorption)** photons at the appropriate frequency/energy

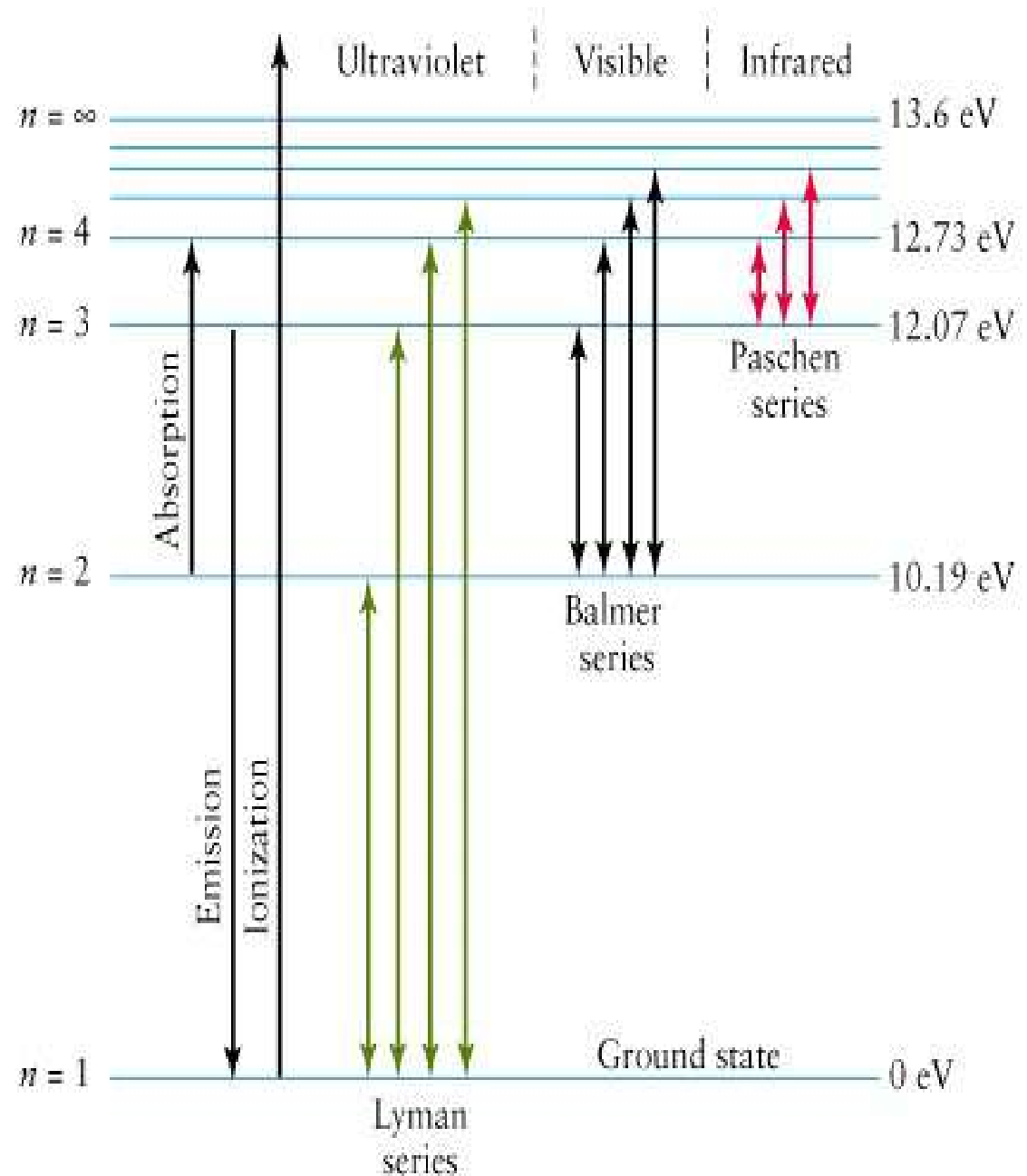
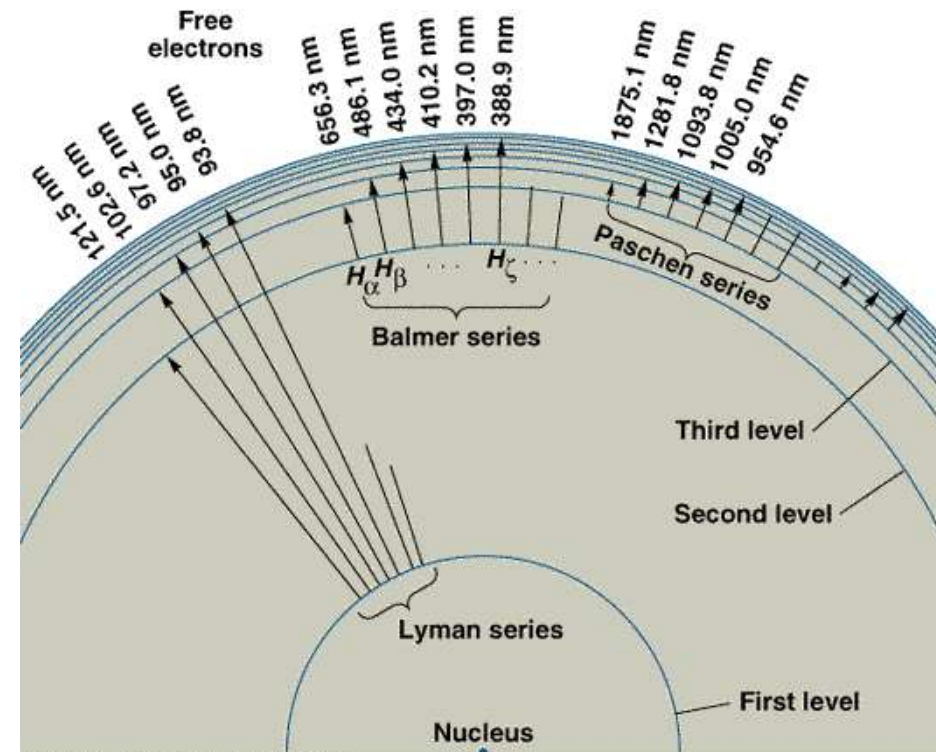
$U \rightarrow \infty$ determines the **binding energy** of a given L status (quantum number)

Based on the main quantum number various series are defined:

Lyman ($n=1$, UV)

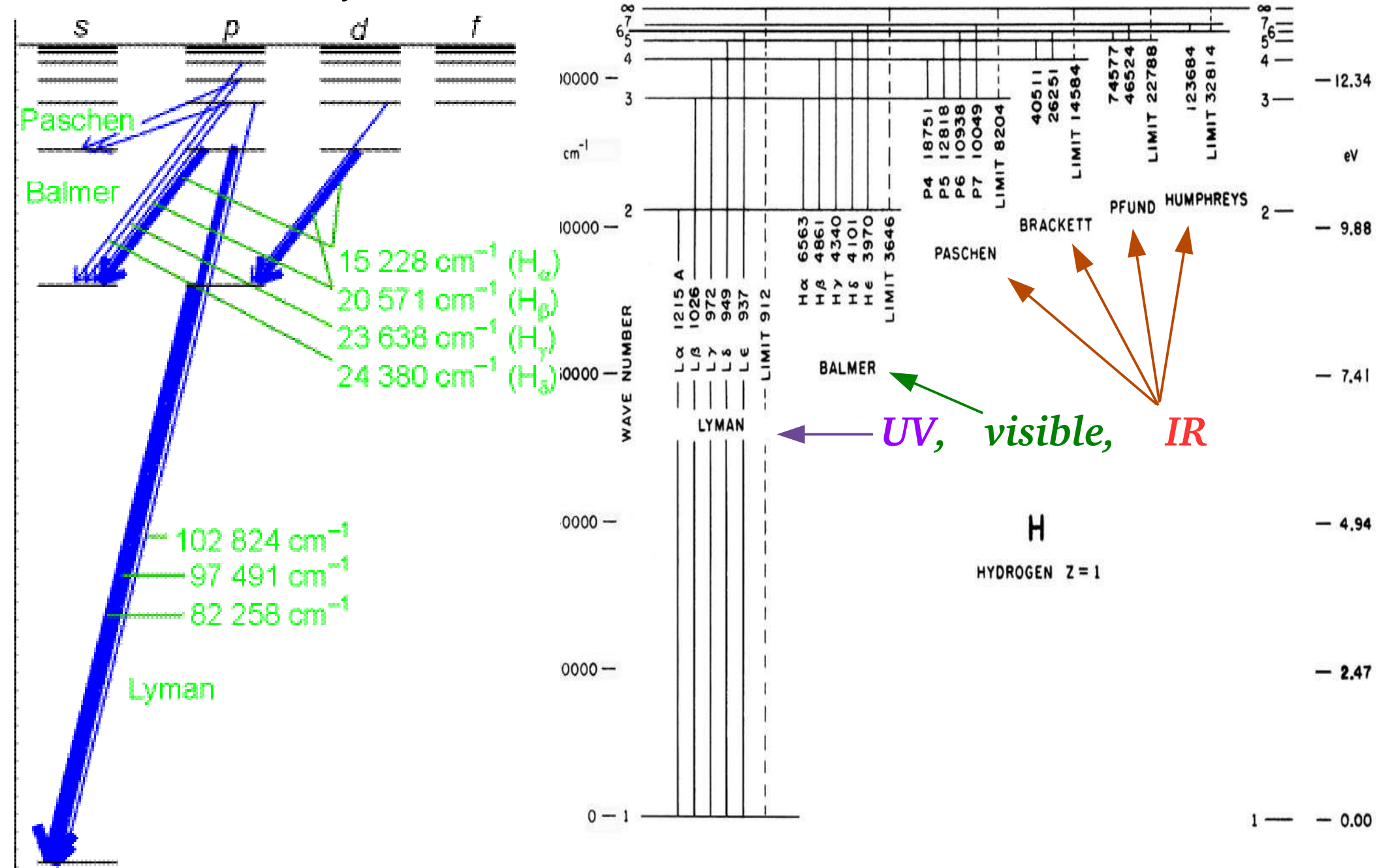
Balmer ($n=2$, visible)

Paschen ($n=3$, IR) ...

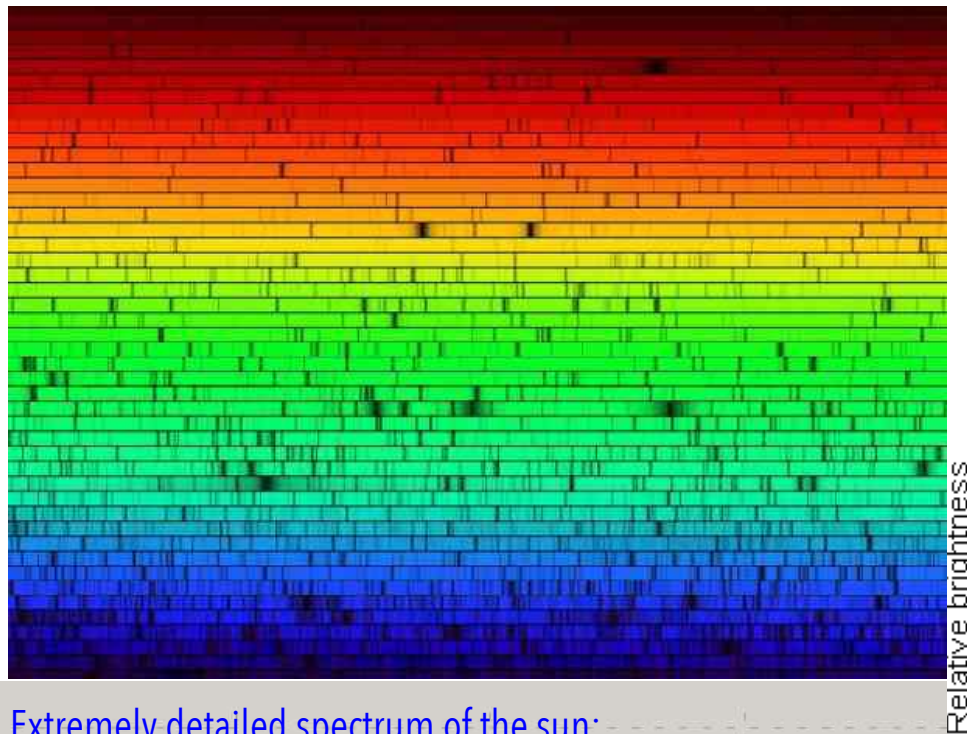
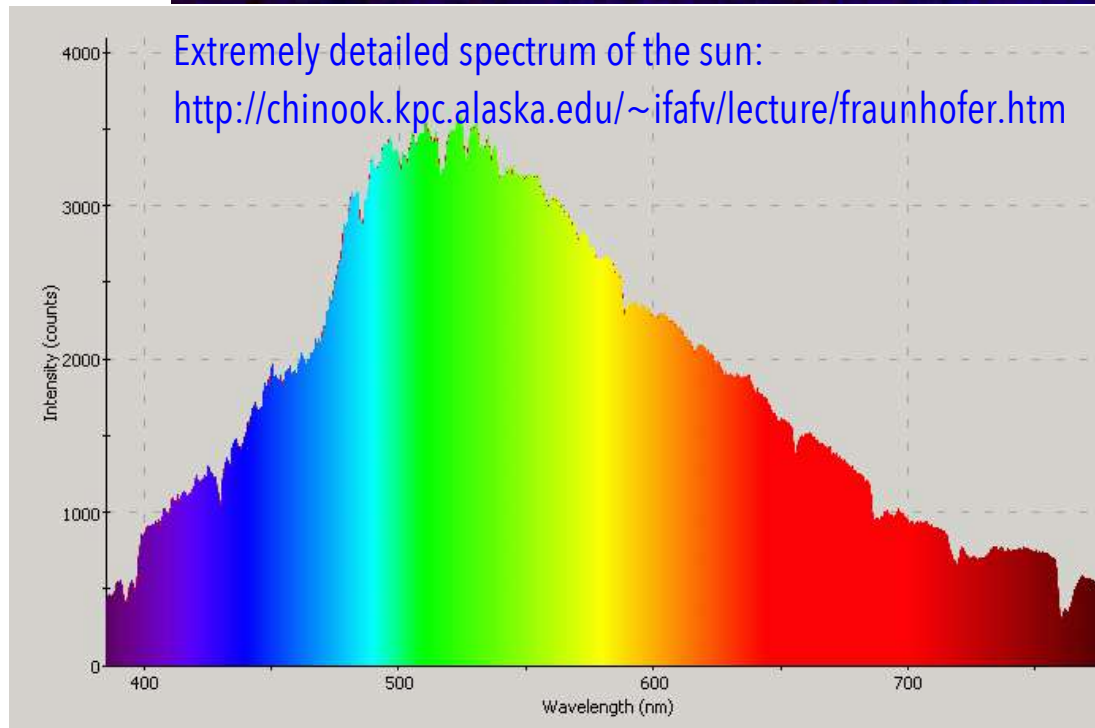
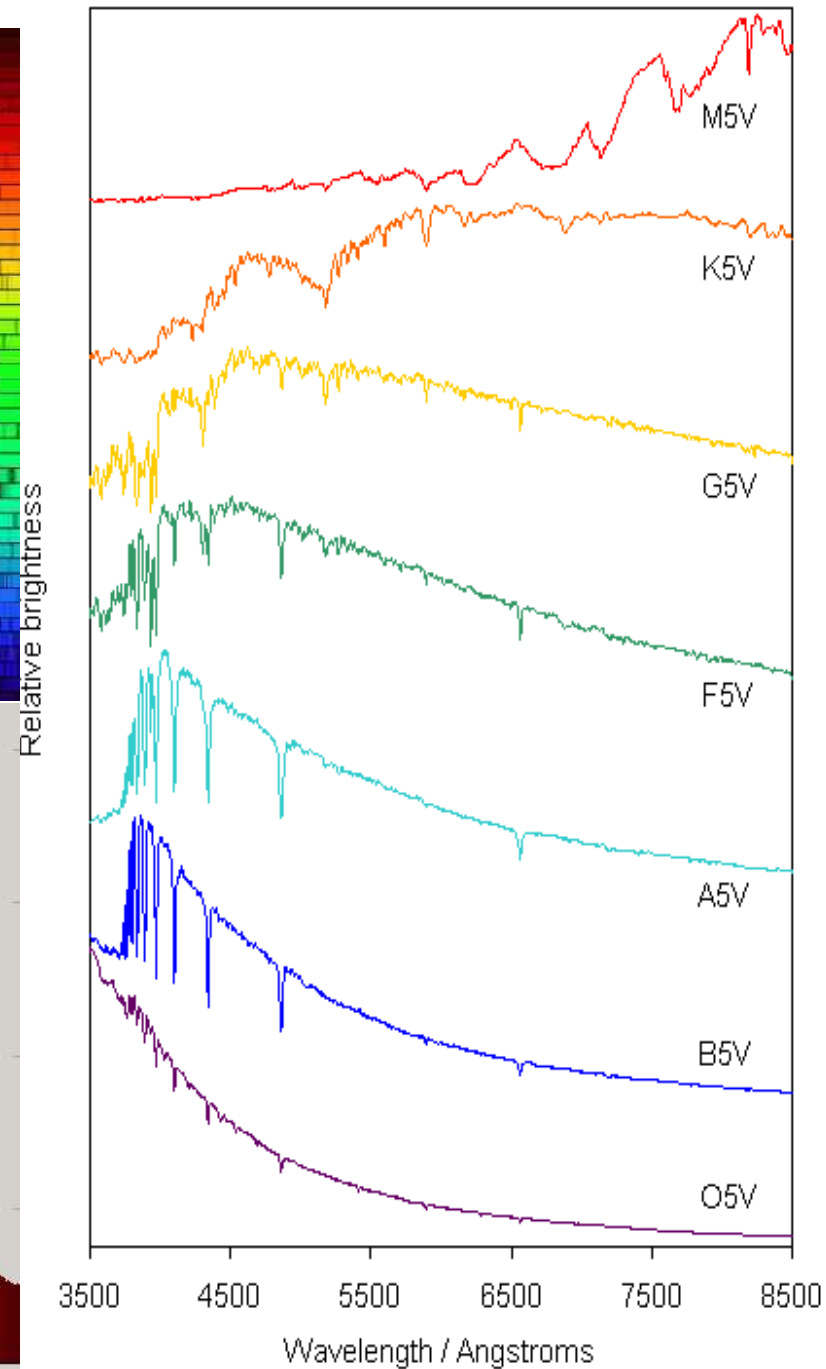


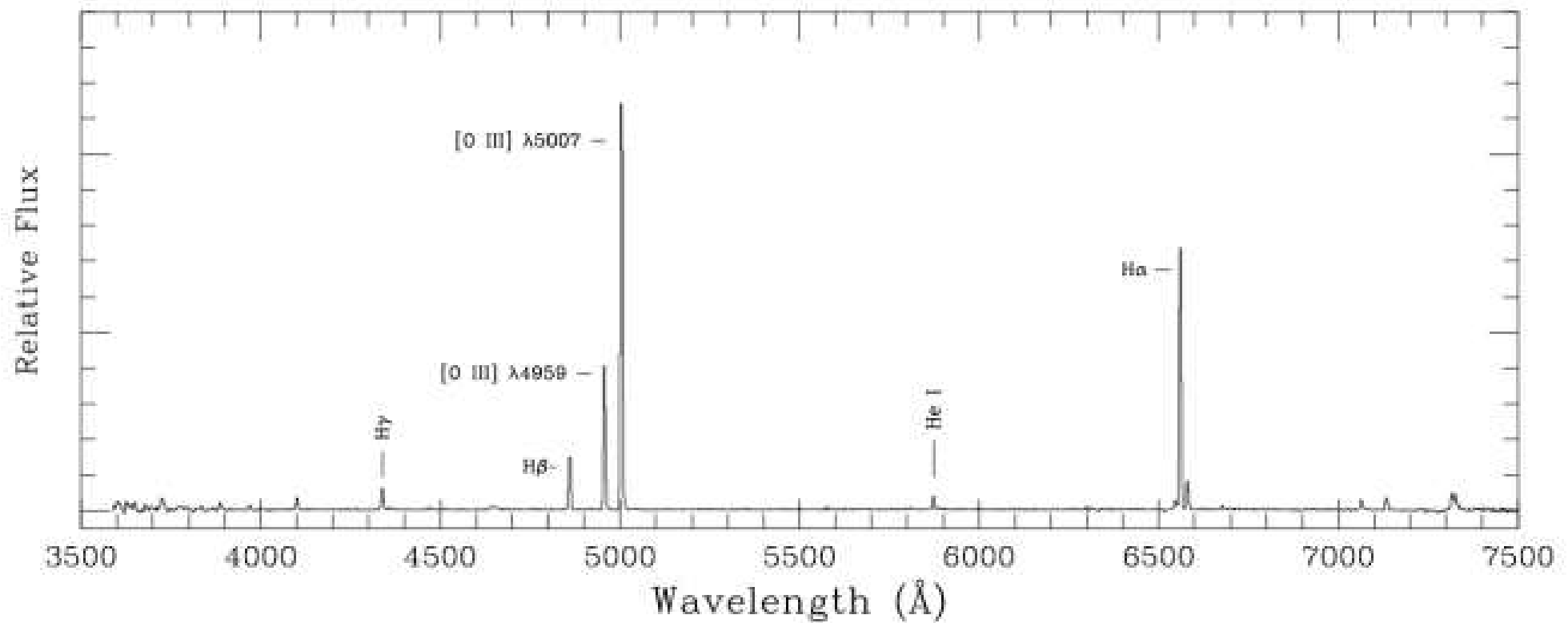


transitions obey to well known **selection rules**: Δn , Δl , Δm ,



There is a background made of BB continuum





> No continuum photons: nonetheless, we have beautiful optical images!

> Typical of gaseous environments like Planetary Nebulae and HII regions



Molecular spectra (0):

Molecules have 3-D structures capable to oscillate around the equilibrium distance (**vibrations**), and may also **change their rotational** energy between adjacent levels defined by quantum mechanics.

Animations @ https://en.wikipedia.org/wiki/Molecular_vibration

Molecules are mostly found in low temperature (T from 10s to a few 100s °K) and high density ($n > 10^2 \text{ cm}^{-3}$) regions in the ISM in spiral galaxies.

They can have electronic transitions in any of the atoms in the molecule (see earlier), not considered here (small differences in the binding energy of the electron)

2 Atoms		3 Atoms		4 Atoms	5 Atoms	6 Atoms	7 Atoms
CH	CP	H ₂ O	N ₂ O	NH ₃	HC ₃ N	CH ₃ OH	CH ₃ CHO
CN	NH	HCO ⁺	MgCN	H ₂ CO	HCOOH	CH ₃ CN	CH ₃ CCH
CH ⁺	SiN	HCN	H3 ⁺	HNCO	CH ₂ NH	NH ₂ CHO	CH ₃ NH ₂
OH	SO ⁺	OCS	SiCN	H ₂ CS	NH ₂ CN	CH ₃ SH	CH ₂ CHCN
CO	CO ⁺	HNC	AlNC	C ₂ H ₂	H ₂ CCO	C ₂ H ₄	HC ₅ N
H ₂	HF	H ₂ S	SiNC	C ₃ N	C ₄ H	C ₅ H	C ₆ H
SiO	N ₂	N ₂ H ⁺	HCP	HNCS	SiH ₄	CH ₃ NC	c-C ₂ H ₄ O
CS	CF ⁺	C ₂ H	CCP	HOCO ⁺	c-C ₃ H ₂	HC ₂ CHO	CH ₂ CHOH
SO	PO	SO ₂	AlOH	C ₃ O	CH ₂ CN	H ₂ C ₆	C ₆ H ⁻
SiS	O ₂	HCO	H ₂ O ⁺	l-C ₃ H	C ₅	C ₅ S	CH ₃ NCO
NS	AlO	HNO	H ₂ Cl ⁺	HCNH ⁺	SiC ₄	HC ₃ NH ⁺	HC ₅ O
C ₂	CN ⁻	HCS ⁺	KCN	H ₃ O ⁺	H ₂ CCC	C ₅ N	
NO	OH ⁺	HOC ⁺	FeCN	C ₃ S	CH ₄	HC ₄ H	
HCl	SH ⁺	SiC ₂	HO ₂	c-C ₃ H	HCCNC	HC ₄ N	
NaCl	HCl ⁺	C ₂ S	TiO ₂	HC ₂ N	HNCCC	c-H ₂ C ₃ O	
AlCl	SH	C ₃	CCN	H ₂ CN	H ₂ COH ⁺	CH ₂ CNH	
KCl	TiO	CO ₂	SiCSi	SiC ₃	C ₄ H ⁻	C ₅ N ⁻	
AlF	ArH ⁺	CH ₂	S ₂ H	CH ₃	CNCHO	HNCHCN	
PN	NS ⁺	C ₃ O	HCS	C ₃ N ⁻	HNCNH	SiH ₃ CN	
SiC		MgNC	HSC	PH ₃	CH ₃ O		
		NH ₂	NCO	HCNO	NH ₃ D ⁺		
		NaCN		HO-CN	H ₂ NCO ⁺		
				HSCN	NCCNH ⁺		
				HOOH	CH ₃ Cl		
				l-C ₃ H ⁺			
				HMgNC			
				HCCO			
				CNCN			

8 Atoms	9 Atoms	10 Atoms	11 Atoms	12 Atoms	13 Atoms	Fullerenes
HCOOCH ₃	CH ₃ OCH ₃	(CH ₃) ₂ CO	HC ₉ N	C ₆ H ₆	c-C ₆ H ₅ CN	C ₆₀
CH ₃ C ₃ N	CH ₃ CH ₂ OH	HO(CH ₂) ₂ OH	CH ₃ C ₆ H	n-C ₃ H ₇ CN		C ₆₀ ⁺
C ₇ H	CH ₃ CH ₂ CN	CH ₂ CH ₂ CHO	CH ₃ CH ₂ OCHO	i-C ₃ H ₇ CN		C ₇₀
CH ₃ COOH	HC ₇ N	CH ₃ C ₅ N	CH ₃ COOCH ₃			
H ₂ C ₆	CH ₃ C ₄ H	CH ₃ CHCH ₂ O				
CH ₂ OHCHO	C ₈ H	CH ₃ OCH ₂ OH				
HC ₆ H	CH ₃ CONH ₂					
CH ₂ CHCHO	C ₈ H ⁻					
CH ₂ CCHCN	CH ₂ CHCH ₃					
NH ₂ CH ₂ CN	CH ₃ CH ₂ SH					
CH ₃ CHNH	HC ₇ O					
CH ₃ SiH ₃						

McGuire, 2018, ApJS, 239, 17

2018 Census of Interstellar, Circumstellar, Extragalactic, Protoplanetary Disk, and Exoplanetary Molecules

McGuire, 2018, ApJS, 239, 17

2018 Census of Interstellar, Circumstellar, Extragalactic, Protoplanetary Disk, and Exoplanetary Molecules



Molecular spectra (1):

Vibrational transitions ($\sim 0.1 - 0.01 \text{ eV} \rightarrow \text{IR}$)

$n = 0, 1, 2 \dots$ is the vibrational quantum number

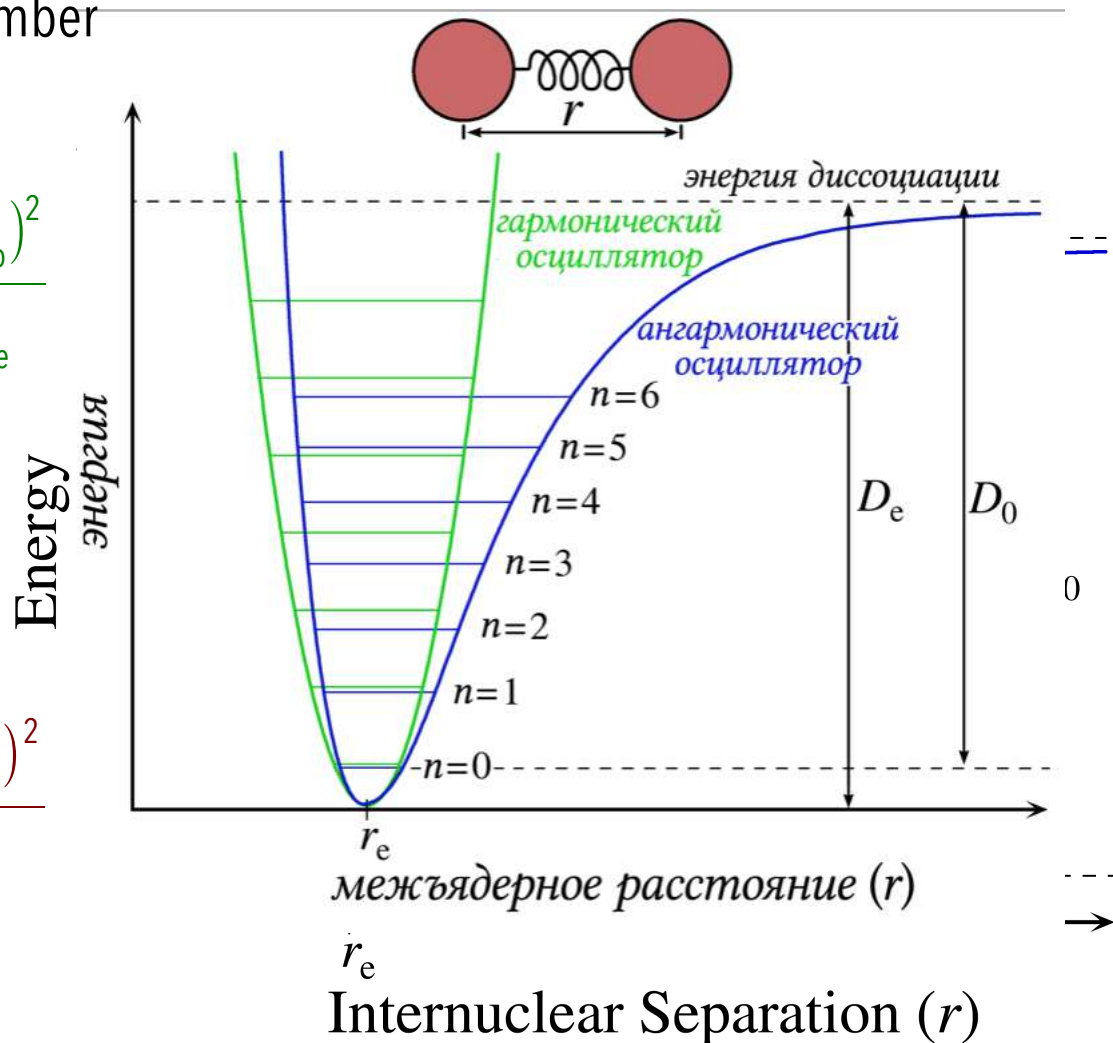
The energy of a given level is

$$E_n = \left(n + \frac{1}{2}\right) h \nu_0 - \left(n + \frac{1}{2}\right)^2 \frac{(h \nu_0)^2}{4D_e}$$

Transitions are allowed for $\Delta n = \pm 1$

Difference between two adjacent levels is

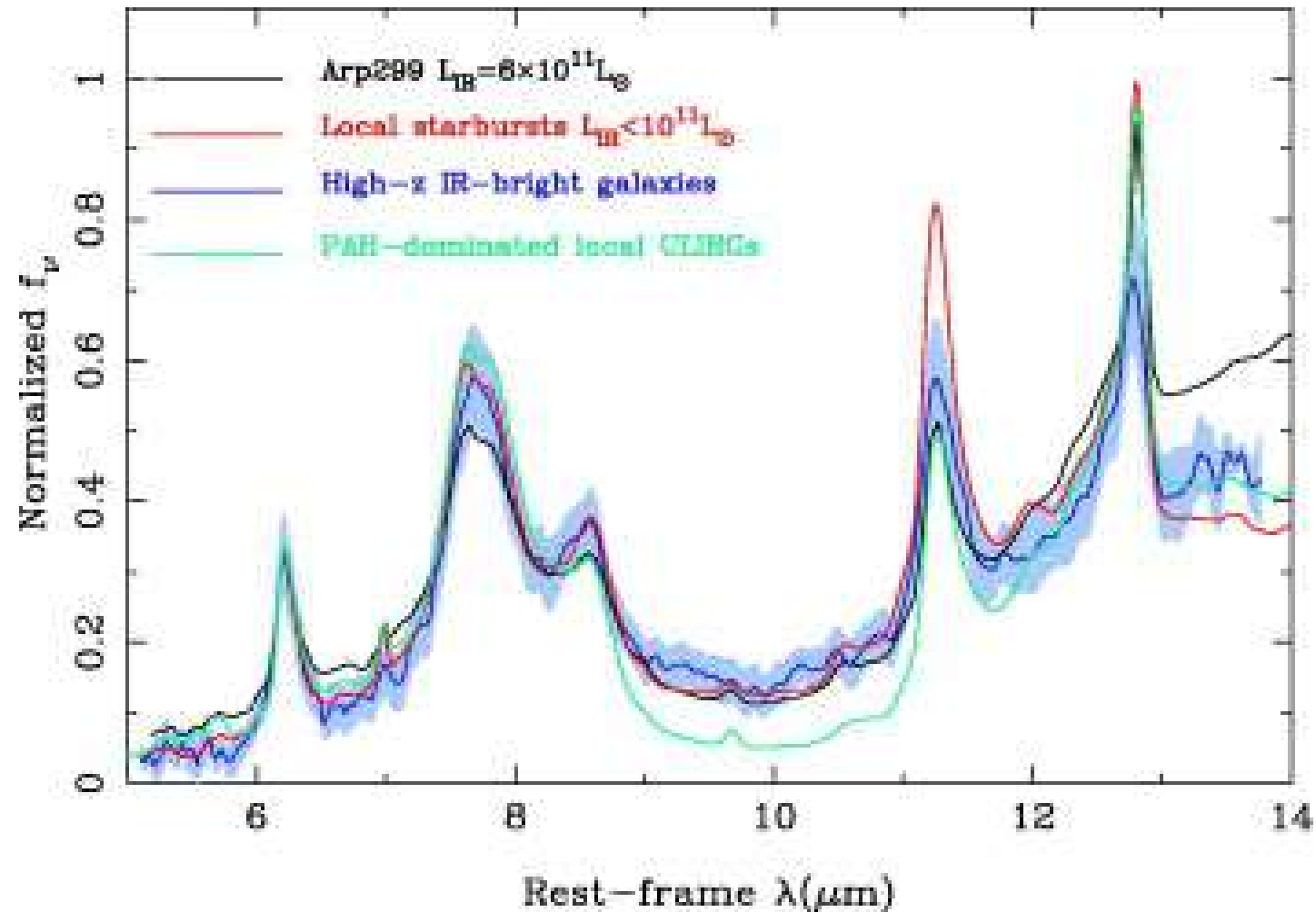
$$E_{n+1} - E_n = h \nu_0 - (n+1) \frac{(h \nu_0)^2}{2D_e}$$



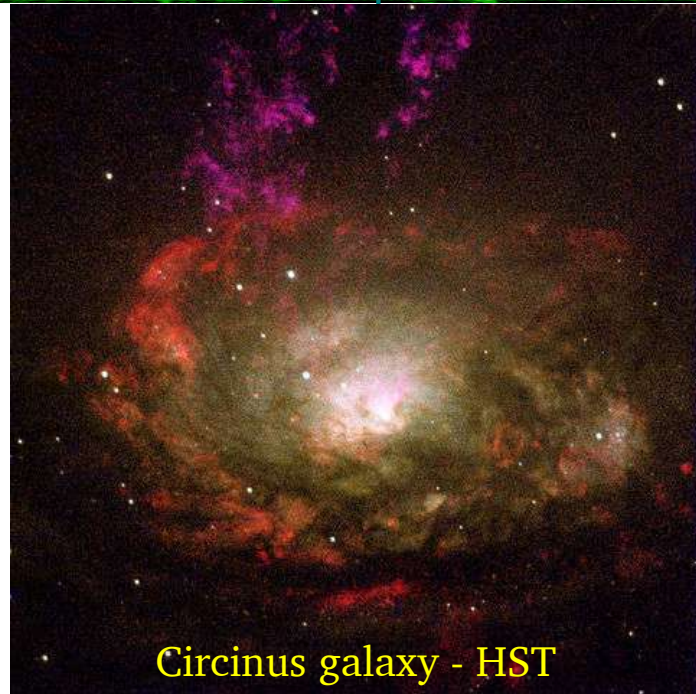
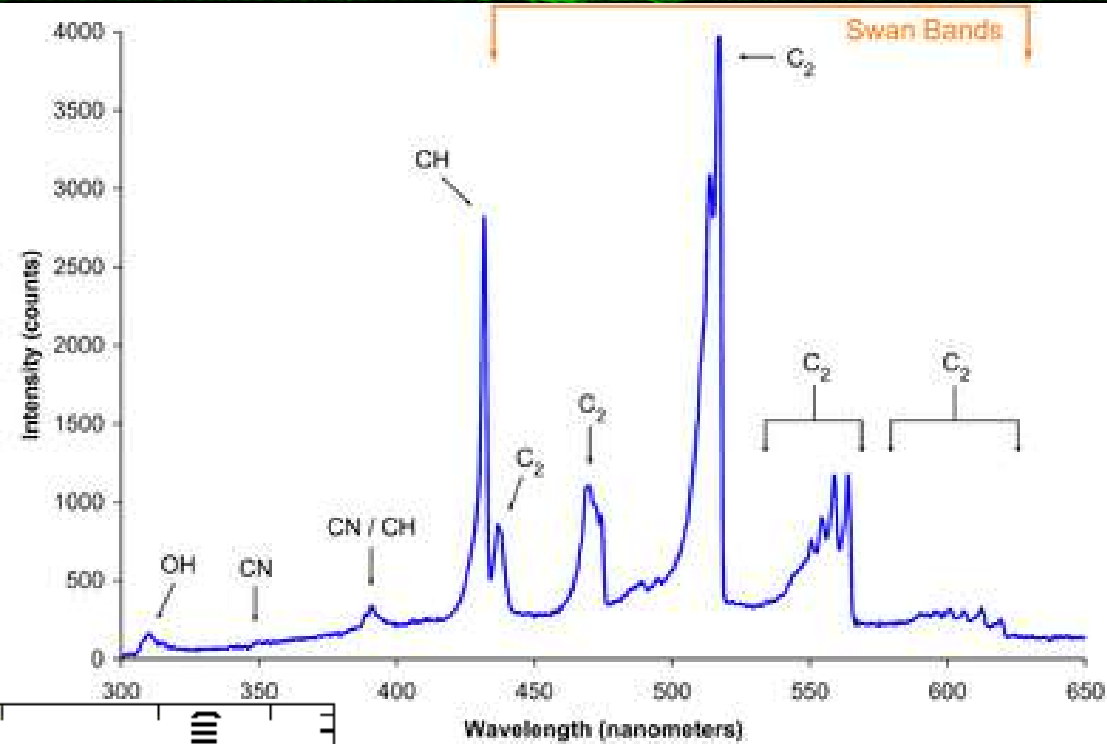
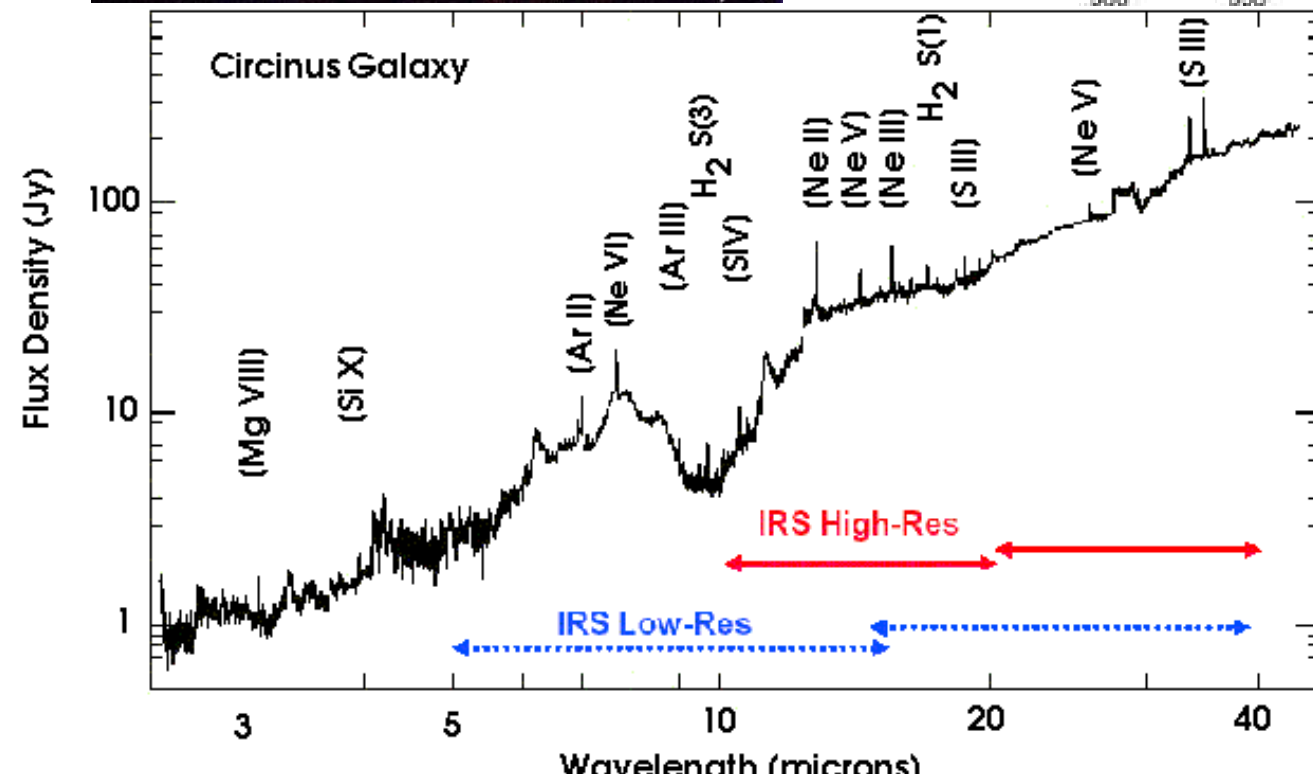
FYI,
Computations
never asked

Molecular spectra (1b): NIR from vibrational transitions

NIR composite spectra for various astrophysical bodies



Molecular spectra (1c): NIR & MIR, from vibrational transitions

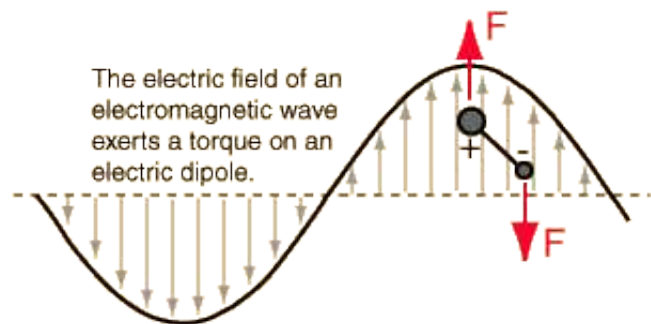
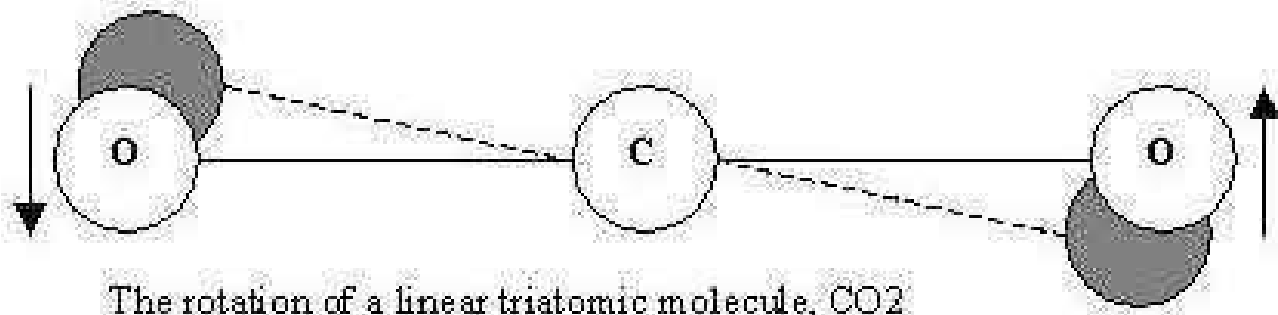
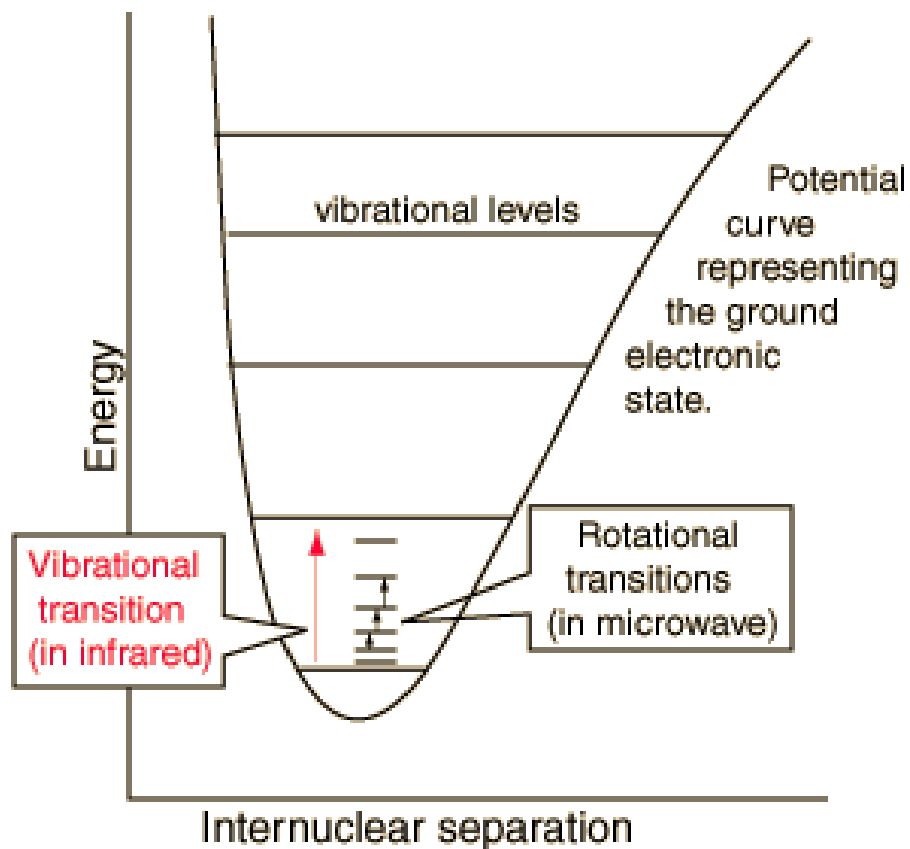




Molecular spectra (2):

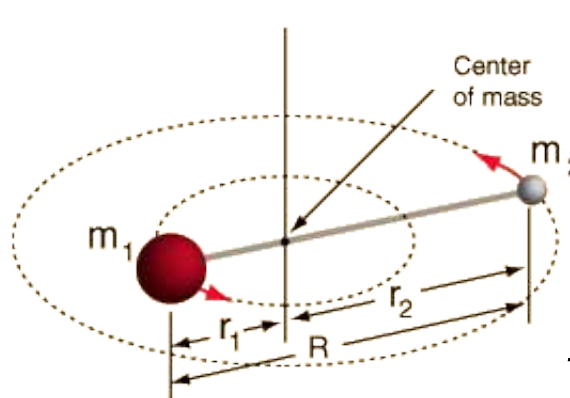
rotational ($\sim \text{meV}$, submm, mm & cm wavelengths)

rotational levels lie within vibrational levels



The rotational quantum number **J** is related to the moment of inertia **I**, to the distance

r and to the rotational energy **E_{rot}**:

$$E_{\text{rot}} = \frac{8\pi^2 I (r_1 + r_2)^2 J(J+1)}{h^2}$$


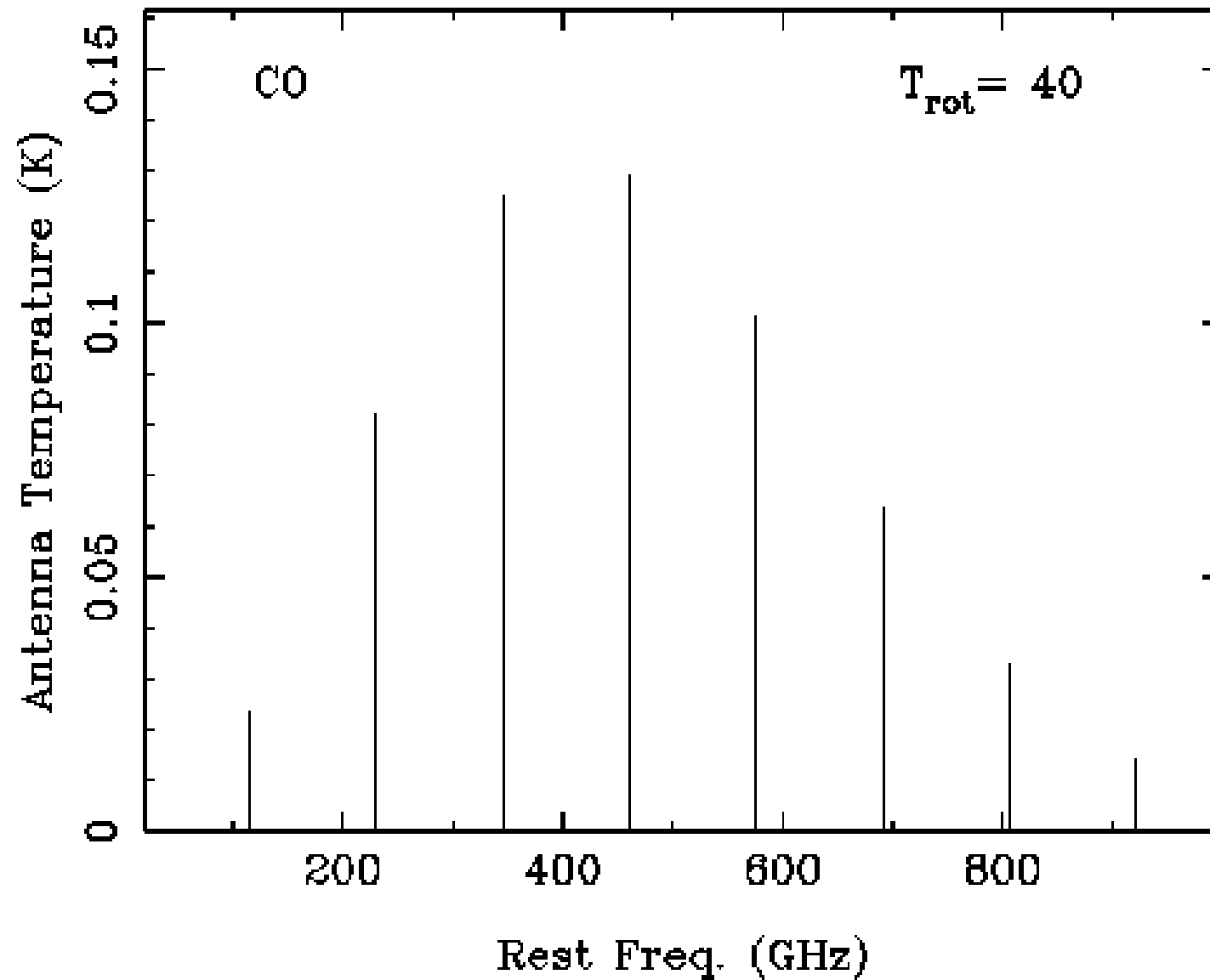
FYI,
Computations
never asked

$$I = m_1 r_1^2 + m_2 r_2^2$$

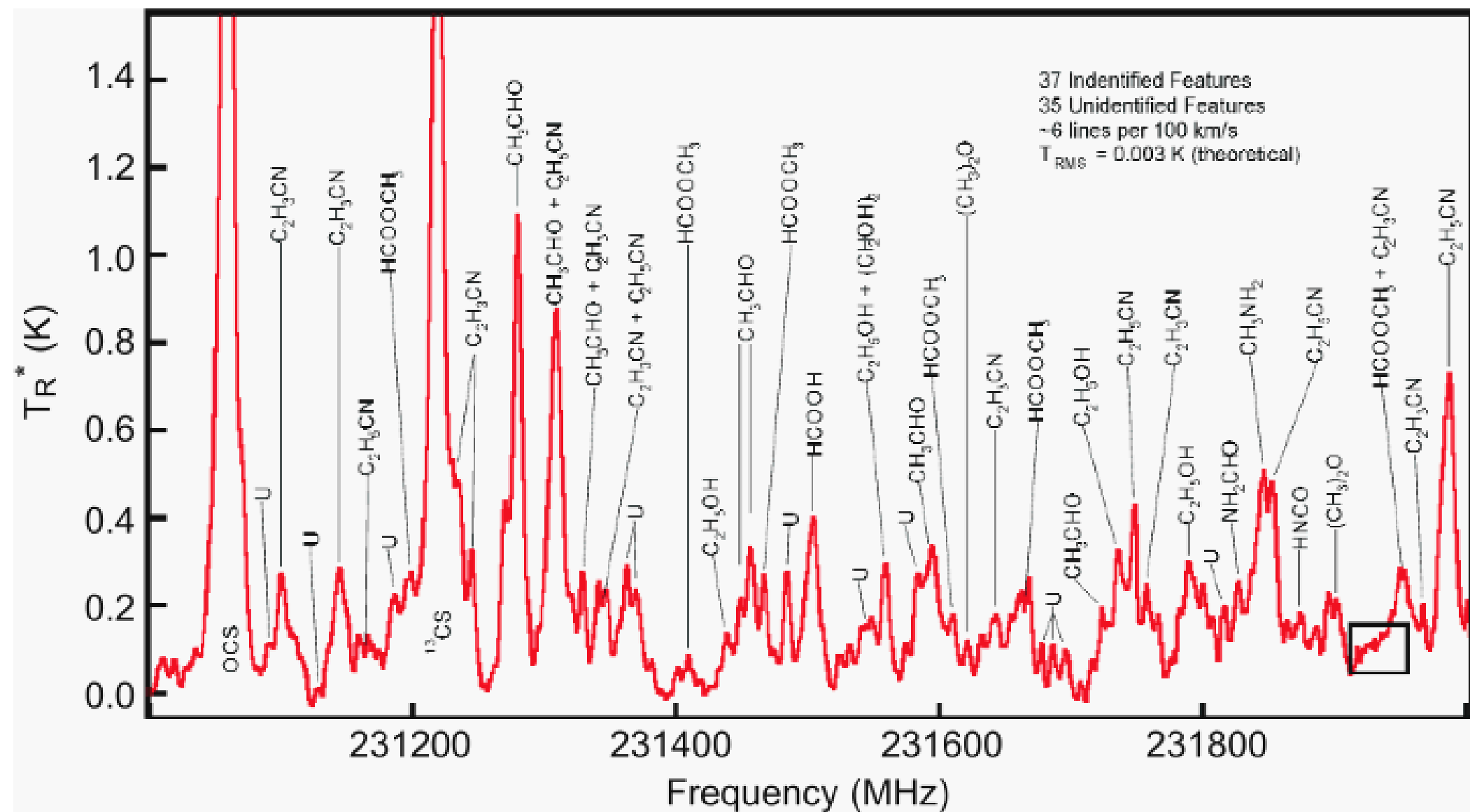


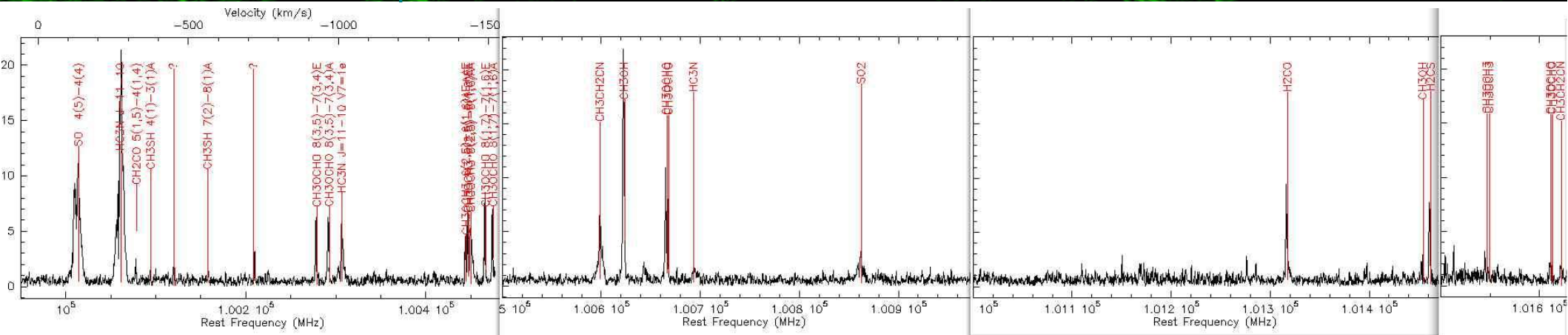
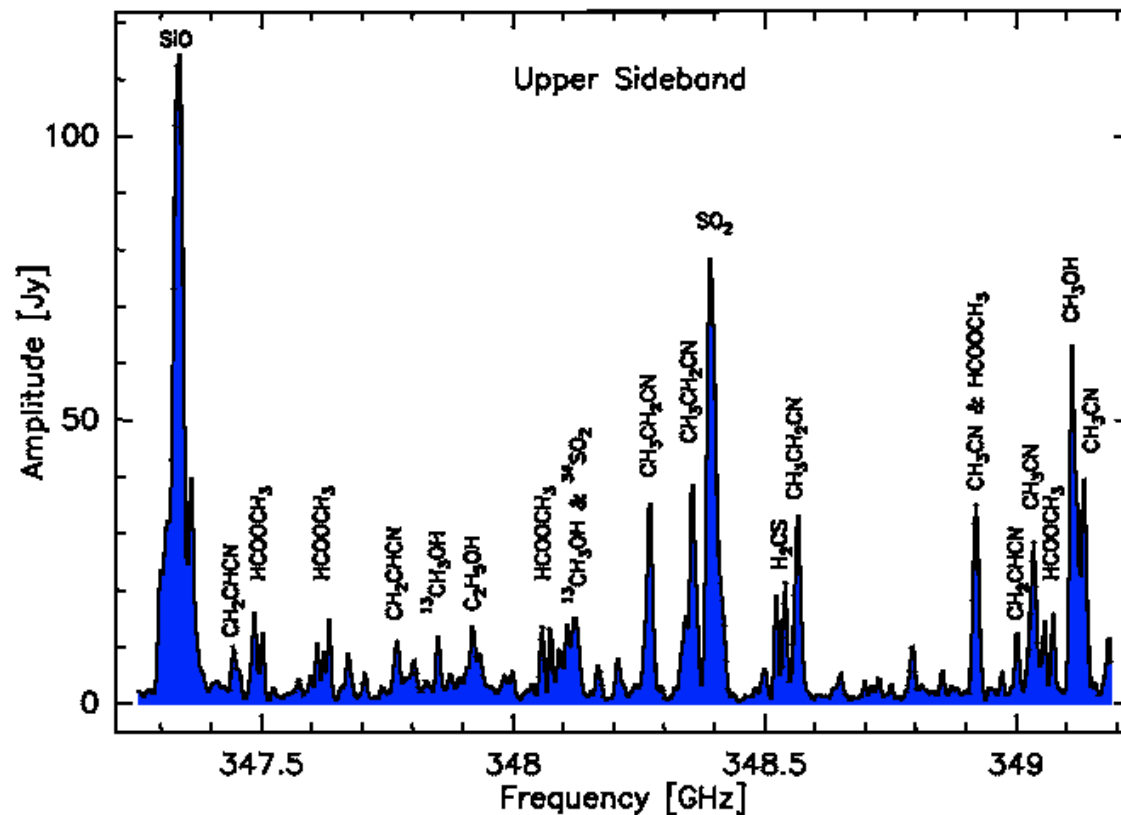
Molecular spectra (2b):

CO emission in a dense, cold cloud (a GMC!)



Molecular spectra (2c):

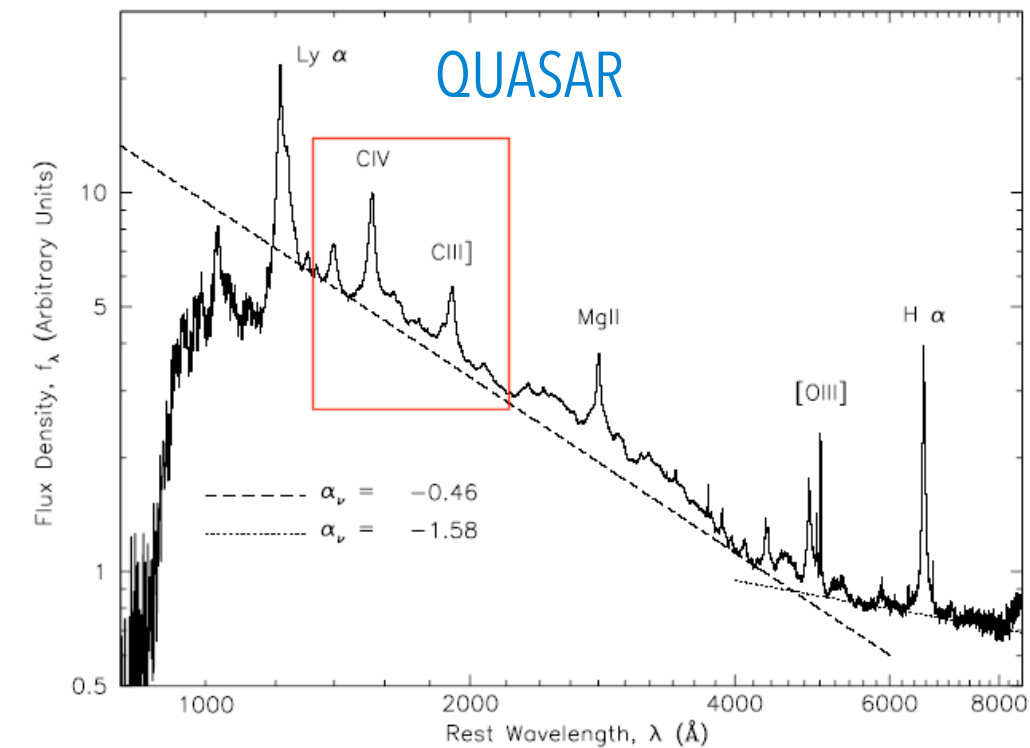
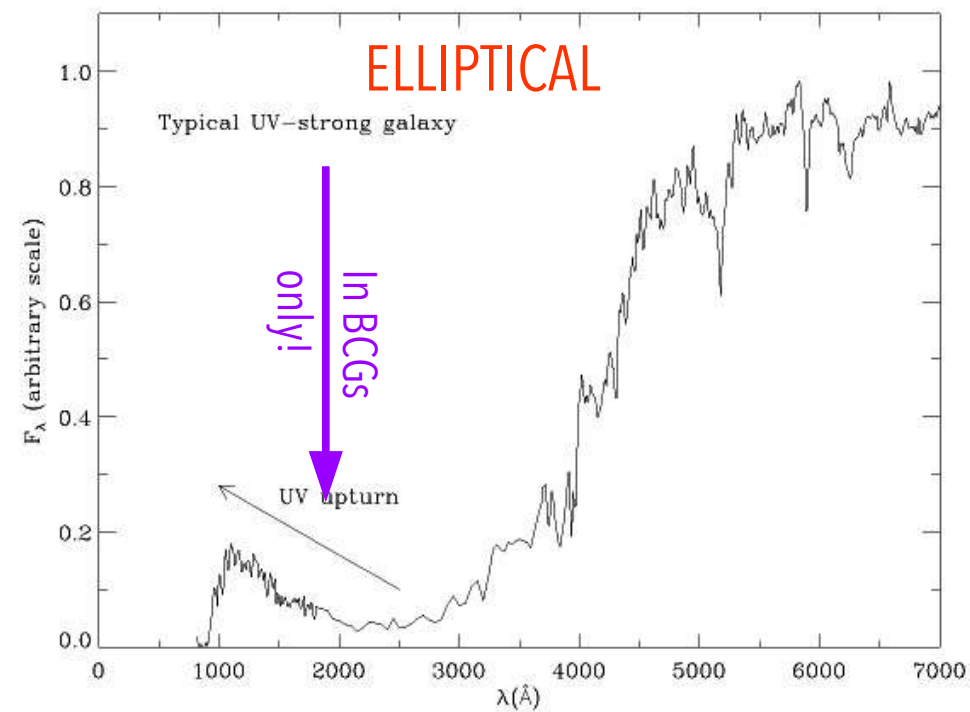




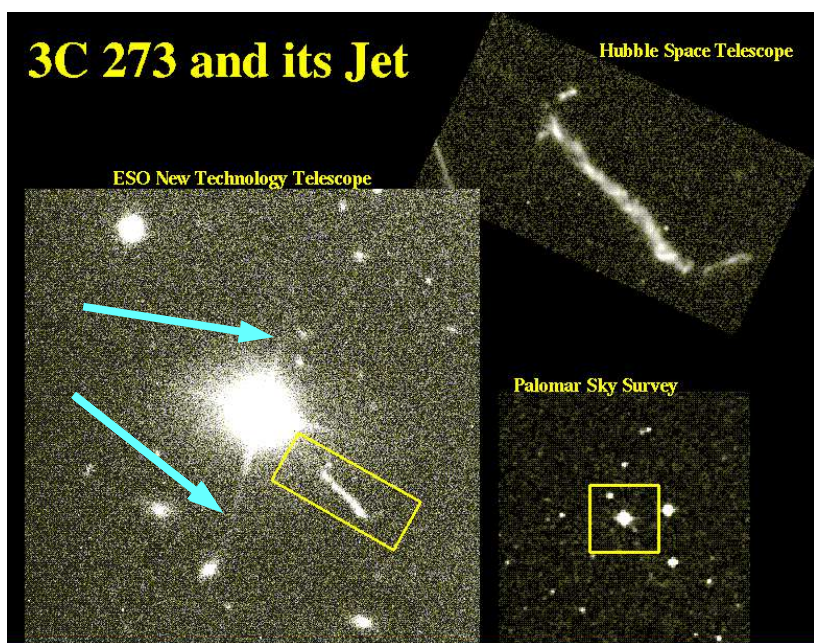
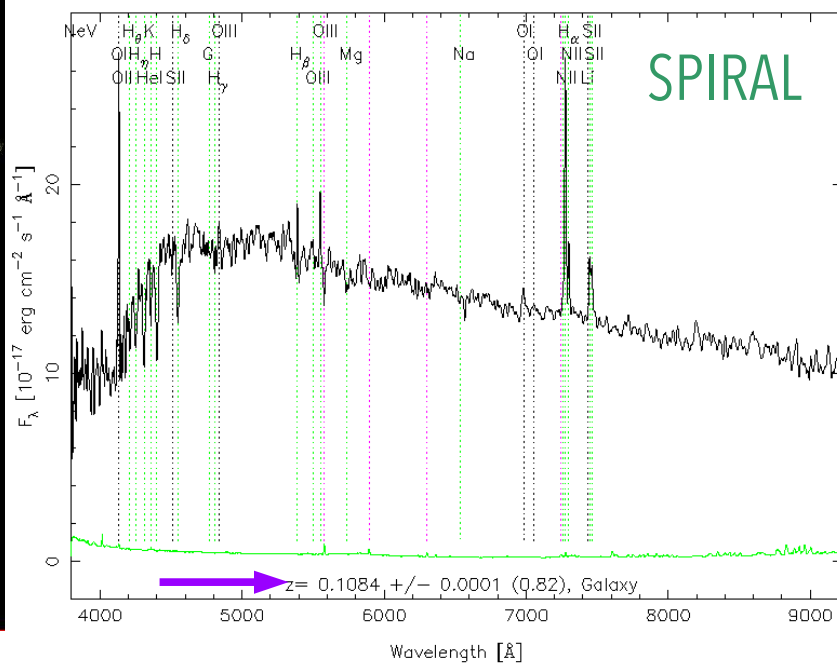
The hot core in Orion KL (high mass SFR)



Optical (& UV) spectra of galaxies



RA=13.78112, DEC=-0.22220, MJD=52883, Plate=1496, Fiber= 76





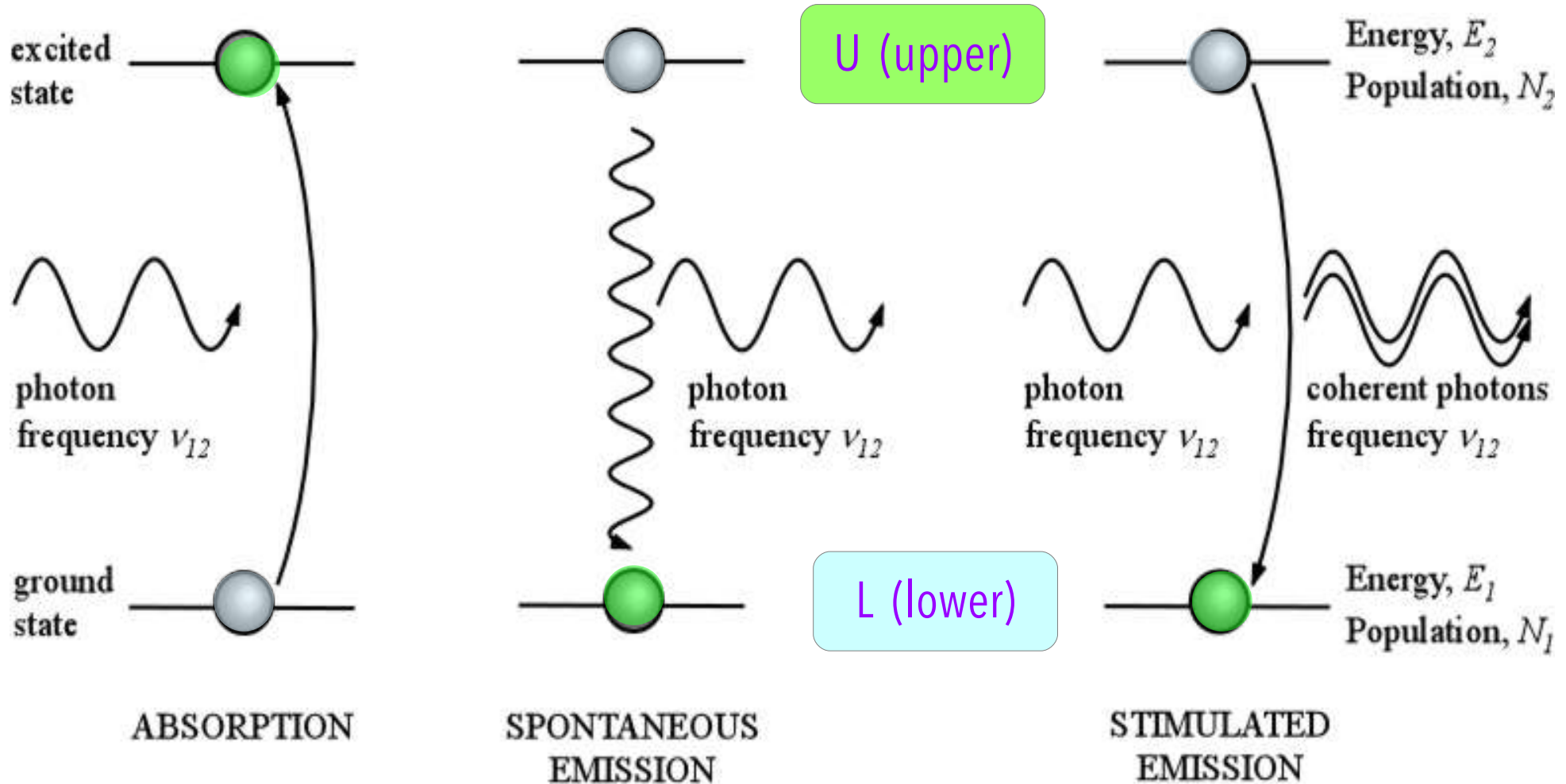
Atomic and molecular spectra: observational constraints

- Lines are generally associated to gaseous (plasma) fluids
- Lines appear either in emission or in absorption (* a background source is required!)
- Lines are not monochromatic
- Lines can be (red – blue) shifted
- Various elements may appear in the same spectrum (i.e. region ?)
Projection effects ?
Intrinsic association ? (In that case physical conditions have to be consistent!)
- Are line ratios a diagnostic for temperature?



Einstein's coefficients (1)

They are used to describe radiative interactions involving bound-bound electron transitions, on the basis of quantum mechanics. Simplest example: an atom with two energy levels only (approx)



Transition energy, $E_{12} = E_2 - E_1 = h\nu_{12}$

**EINSTEIN A & B
COEFFICIENTS**



Einstein's coefficients (2)

Given two atomic levels U and L , (with $U > L$) the following definitions hold:

Spontaneous emission:

A_{UL} = transition probability per unit time to go from level U to L by **emission** of a photon

Absorption:

$B_{LU}\bar{J}$ = transition probability per unit time to go from level L to U via **absorption** of a photon

Stimulated emission:

$B_{UL}\bar{J}$ = transition probability per unit time $U \rightarrow L$ via **emission** induced by incoming radiation

$\bar{J}$ represents the density of photons at (about) the frequency ν_{UL} so that $h\nu_{UL}$ corresponds to the difference between the energy levels U and L

A_{UL} is given in sec^{-1} units

$\frac{1}{A_{UL}}$ is the mean lifetime of the excited state

The set of emitted photons **IS NOT monochromatic**

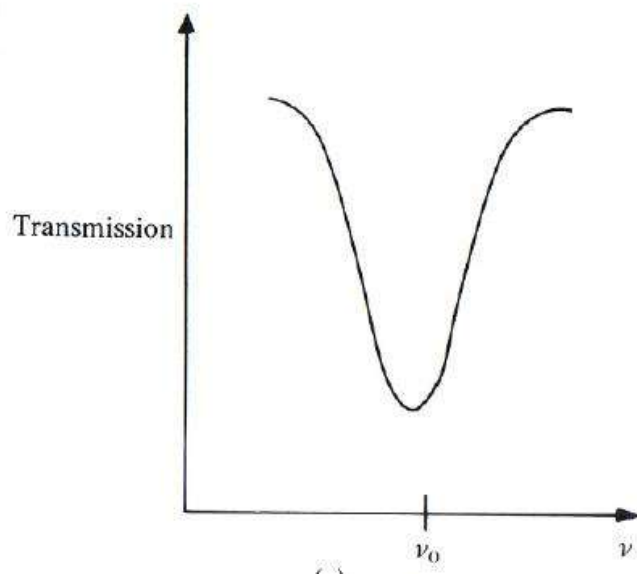
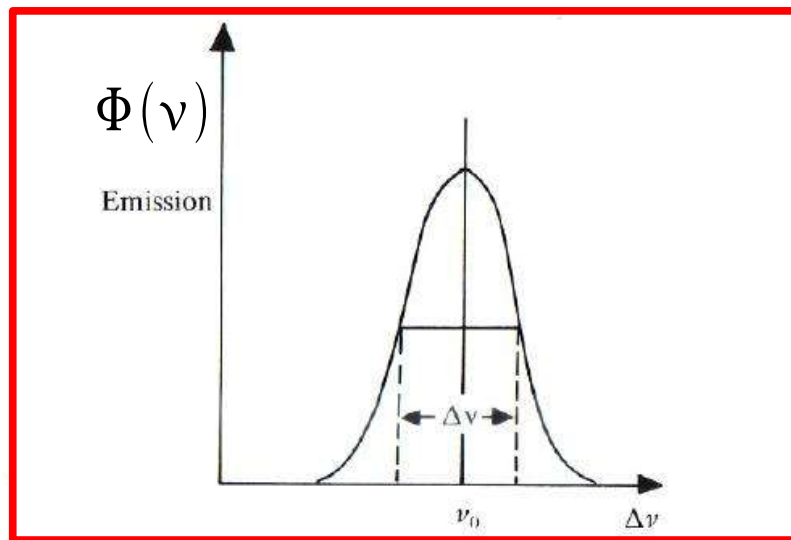
- the energy difference between the two levels is not infinitely sharp.
- described by the **line profile function (LPF)** $\Phi(\nu)$ (sharply?) peaked at the frequency ν_{UL}

$$\int_0^{\infty} \Phi(\nu) d\nu = 1$$

It represents **how effectively photons with frequencies around ν_{UL} can cause transitions**

$$\bar{J} \equiv \int_0^{\infty} J(\nu) \Phi(\nu) d(\nu)$$

If $J(\nu)$ changes slowly with ν , then $\Phi(\nu)$ acts like a δ function



Natural (Lorentz) broadening

An atom at an excited state has a finite lifetime $\Delta t \simeq \frac{1}{A_{UL}}$

The energy difference is finite as well $\Delta E = \Delta h \nu$

The decay therefore follows the Heisenberg's Principle in the form $\Delta E \Delta t \geq \frac{h}{2\pi}$

$$\Delta E \Delta t = \Delta h \nu \frac{1}{A_{UL}} \geq \frac{h}{2\pi}$$

$$\Delta \nu \geq \frac{A_{UL}}{2\pi}$$

and this is the **minimum width** allowed for the transition between levels U, L

WARNING: **large**/**small** values of A_{UL} imply **broad**/**narrow** lines

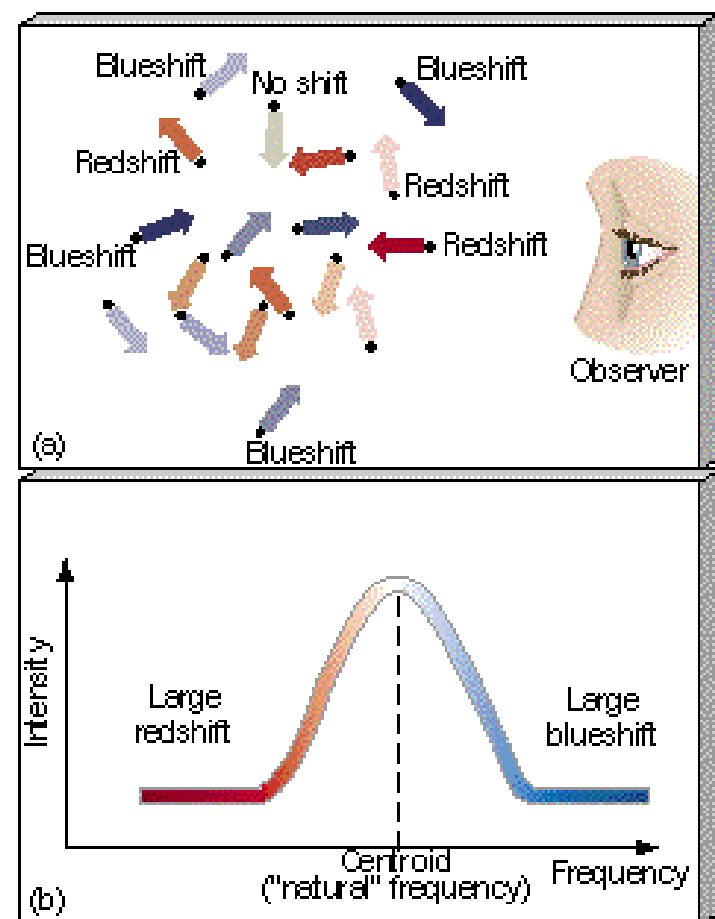
Doppler (thermal) broadening

Atoms are in (thermal) motion wrt the observer and the rest (atom) frame frequencies may be either **red-** or **blue-** shifted. If v_r is the radial velocity

$$\Delta \nu = \nu_{\text{obs}} - \nu_{\text{em}} = \nu_{\text{em}} \frac{v_r}{c}$$

It is possible to derive the radial velocity (**redshift/blueshift**)

$$v_r = c \frac{\nu_{\text{obs}} - \nu_{\text{em}}}{\nu_{\text{em}}} \quad [\text{OK for small } v_r]$$



→ **the shape of the line is modified, the total energy is not**

N.B. The centroid (= "natural frequency") **remains unchanged;**

Only in the case that the **whole cloud is moving, then also the centroid is (Doppler) shifted**



Spectral line broadening (3) – v_r and temperature

In case we are at thermal equilibrium (Maxwell-Boltzmann)

$$N(v_r) dv_r \simeq N_0 v_r^2 e^{\frac{-m_a v_r^2}{2kT}} dv_r \quad \text{where } m_a \text{ is the mass of the atom}$$

$$N(v) dv \simeq N_0 e^{\frac{-m_a c^2}{2kT} \frac{(v_{\text{obs}} - v_{\text{em}})^2}{v_{\text{em}}^2}} dv$$

And in this case the Line Profile Function becomes (if we introduce the **Doppler width**)

$$\Delta v_D = \frac{v_{\text{em}}}{c} \sqrt{\frac{2kT}{m_a}}$$

And the LPF is

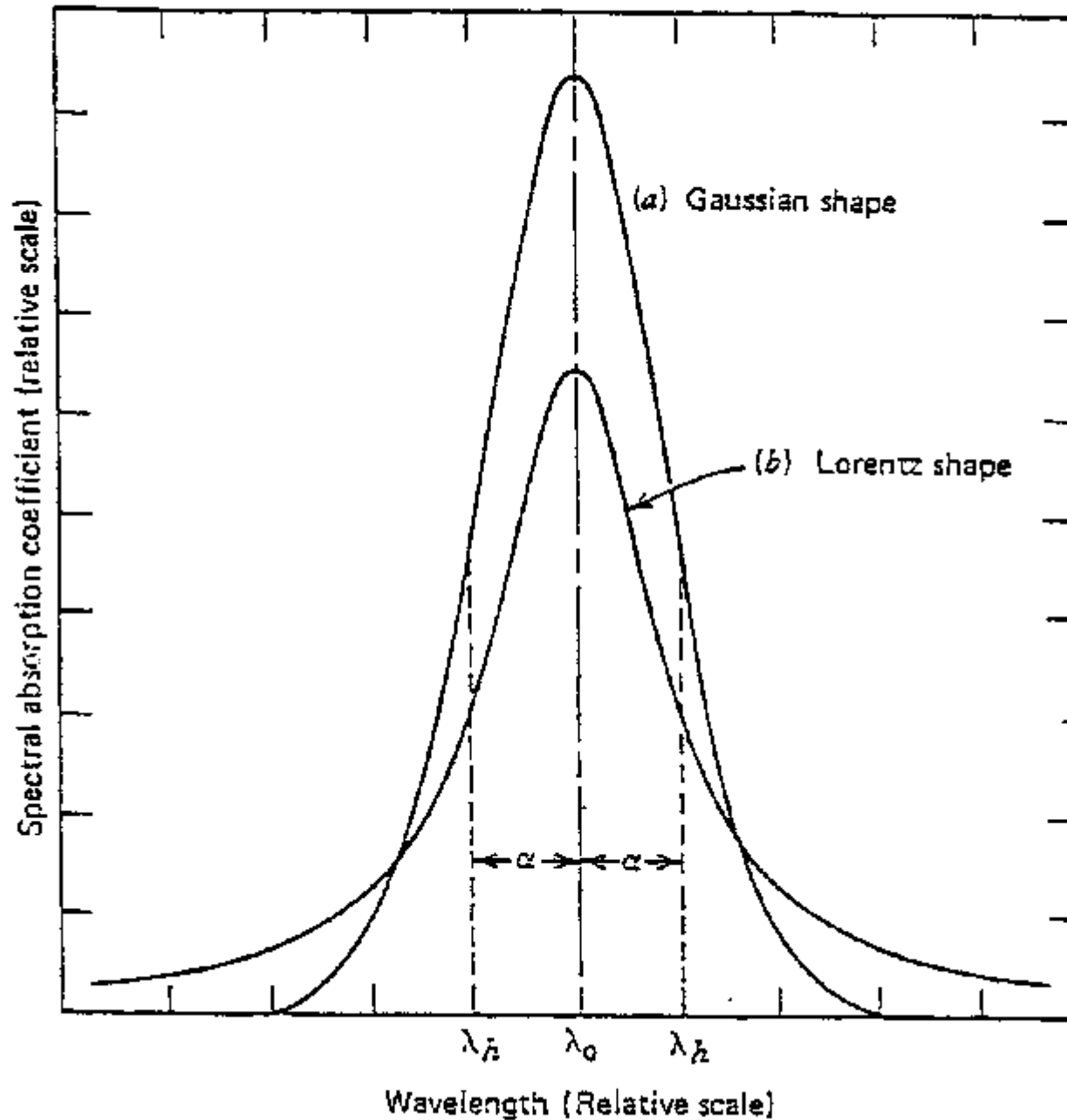
$$\Phi(v) = \left(\Delta v_D \sqrt{\pi} \right)^{-1} \exp \left(- \frac{(v_{\text{obs}} - v_{\text{em}})^2}{(\Delta v_D)^2} \right)$$

Valid also in case of turbulence, once modified the Doppler width by introducing ξ , the rms of turbulent velocities (with Gaussian distribution)

$$\Delta v_D = \frac{v_{\text{em}}}{c} \sqrt{\frac{2kT}{m_a} + \xi^2}$$

FYI,
Computations
never asked

Spectral line broadening (5)

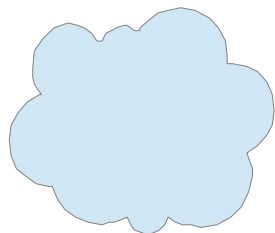
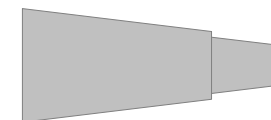


Spectral line shape produced by: (a) Doppler broadening and (b) natural and collision broadening (taken from [Levi \(1968\)](#)).

Test (1) : Do we understand how it works?

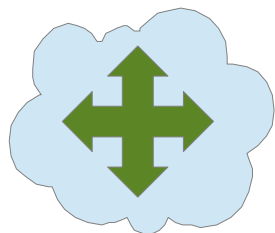
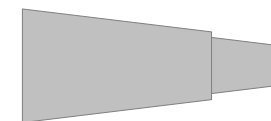
question 1:

(e.g. a standing cloud)



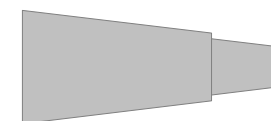
question 2:

(e.g. an expanding cloud)



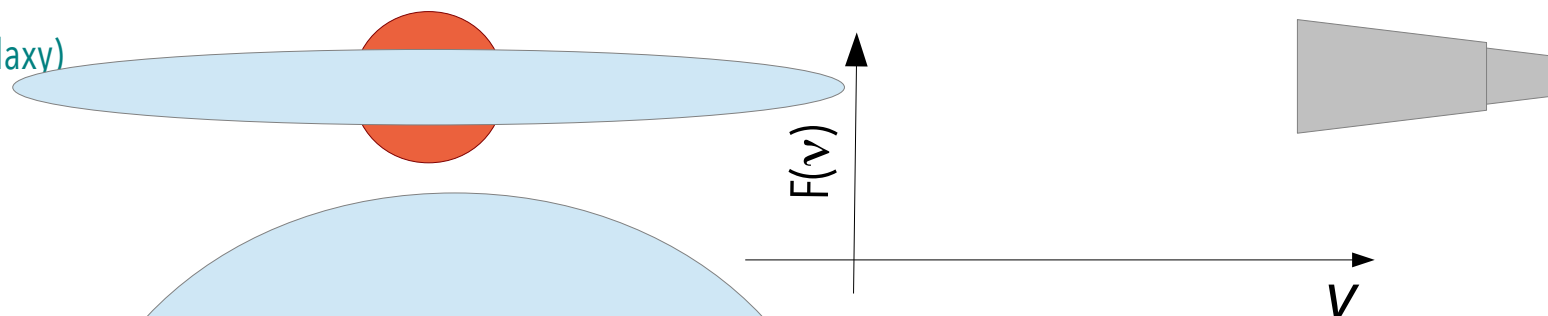
question 3:

(2 clouds orbiting around a point)



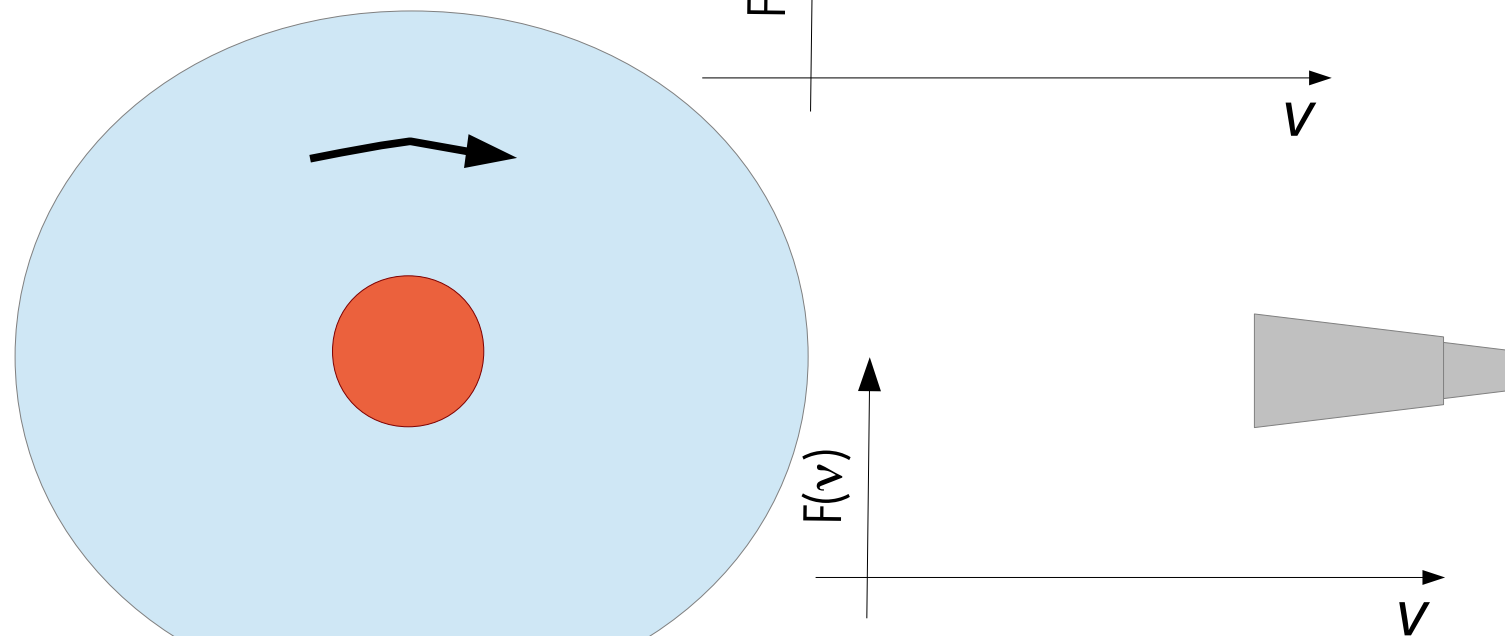
question 4:

(e.g. an edge-on spiral galaxy)



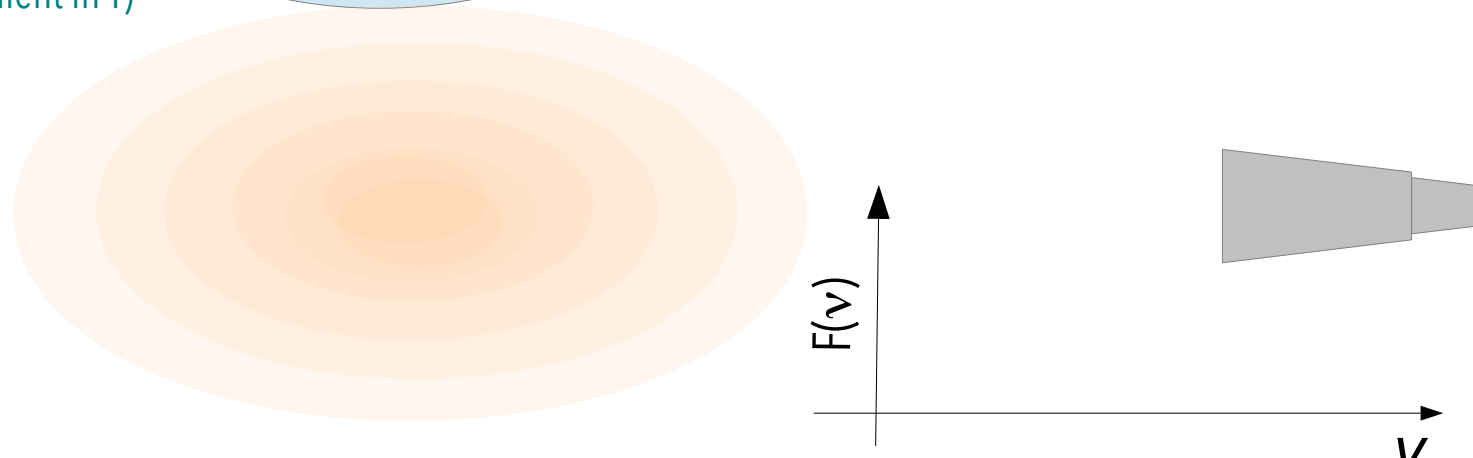
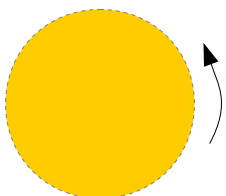
question 5:

(e.g. a face-on spiral galaxy)



question 6:

(e.g. a stratified gas with a gradient in T)



question 7:

(e.g. a rotating star)



Test (+) : Answers

[the layout of this page stands as it is to leave room for individual

answers]

Question 1/4:

Question 2/5:

Question 3/6:

The galactic disk is interpreted as the sum of many clouds, each with the same emission.

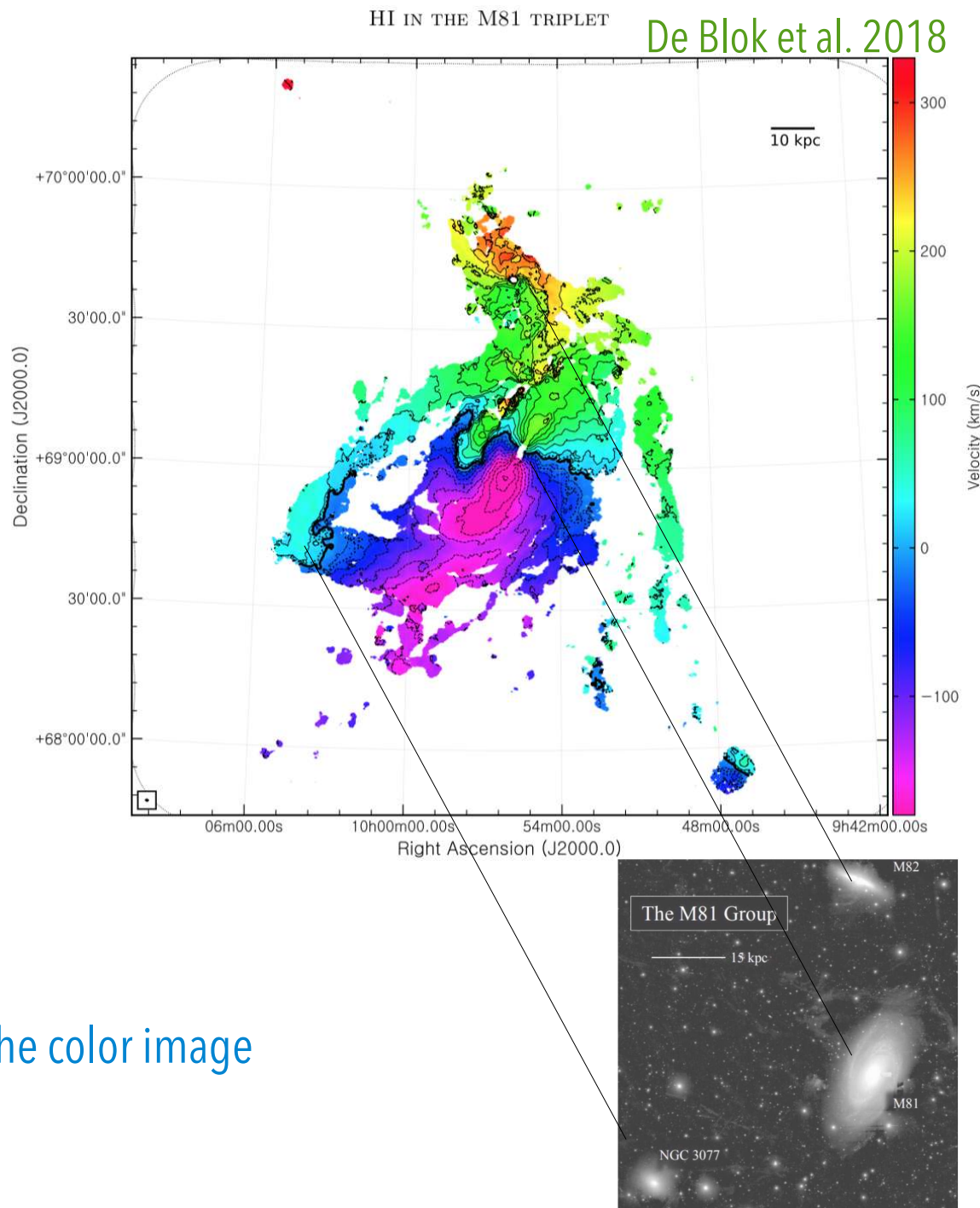
The M81/M82/NGC3077 group

Color image: gas emission

The centroid is shifted according to their v_r
Magenta & Blue regions are approaching
Cyan, Green & Yellow/Orange regions are receding.

Bridges of gas are connecting the 3 galaxies

Bottom right:
optical image (stars) with the same scale as the color image



Visible in continuum from stars

➤ **Origin:**

bound-free transition, electrons from $n=2$ are ionised

➤ **Effect:**

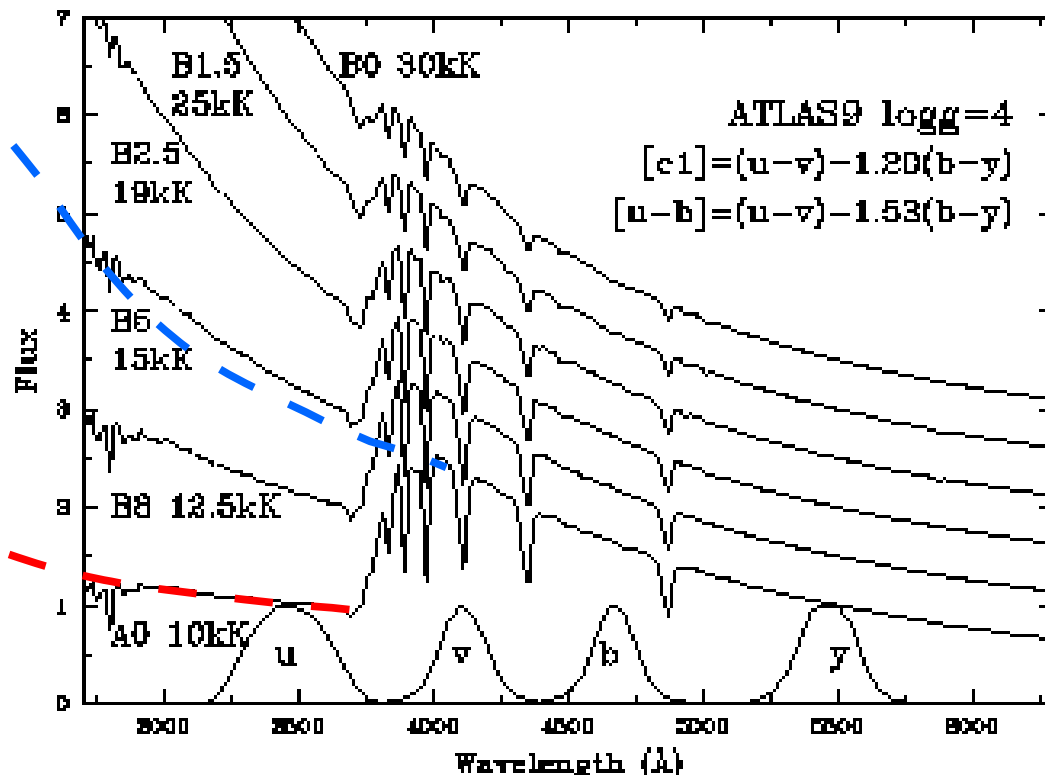
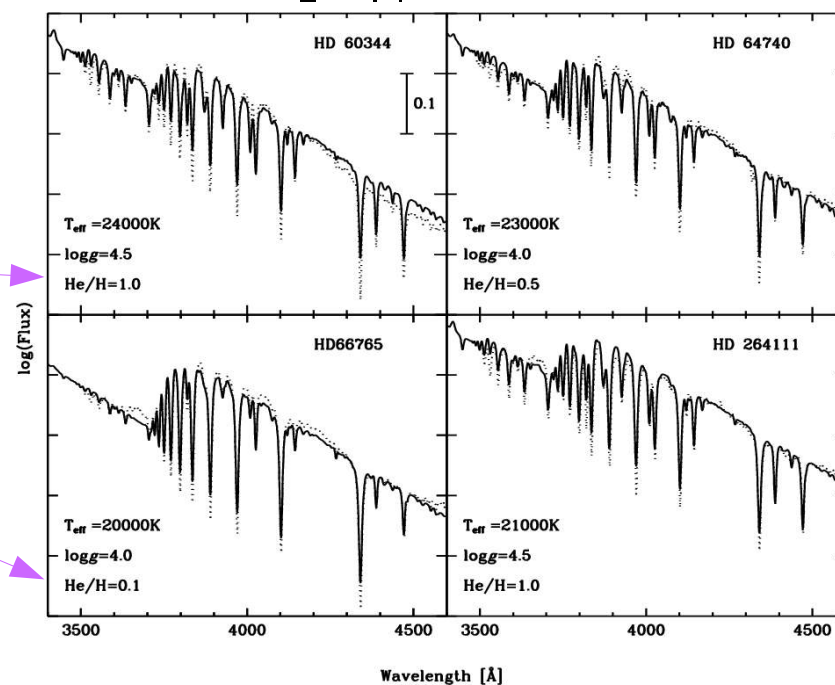
depression of the continuum

➤ **Depends on:**

Density (composition) & Temperature of the region responsible for the absorption (largest in A-type stars)

➤ **Located at:**

The Balmer series limit and higher energies





Atoms as oscillators: electric dipole transitions (2)

Emitted power & **the meaning of A_{ul}**

In classical physics, during dipole transitions, the electron "makes an oscillation" with a given ω :

$$d(t) = x_0 \cos(\omega t) \rightarrow \ddot{d}(t) = -x_0 \omega^2 \cos(\omega t) \Leftrightarrow a(t)$$

$$P(t) = \frac{2e^2}{3c^3} a^2(t) = \frac{2e^2}{3c^3} \omega^4 x_0^2 \cos^2(\omega t)$$

$$\langle P(t) \rangle = \frac{2e^2}{3c^3} \frac{\omega^4 x_0^2}{2} = \frac{32\pi^4}{3c^3} v^4 \left(\frac{e^2 x_0^2}{2} \right) = \frac{32\pi^4}{3c^3} v^4 \langle \ddot{d}_x^2(t) \rangle$$

$$= A_{ul} \cdot h \nu_{ul} \quad (\text{i.e. number of transitions per unit time} \times h \nu_{ul})$$

The line power/luminosity can be interpreted as the result of spontaneous emission processes in case it is the only effective mechanism (multiplied by the number density n_e and the volume V)



The meaning of A_{UL}

example:

$$x_0 = r_0 = 5.3 \cdot 10^{-9} \text{ cm (Bohr radius)}$$

$$\nu_{UL} = 3.3 \cdot 10^{15} \text{ Hz (Lya limit)}$$

$$d = e \frac{r_0}{2} = 1.3 \cdot 10^{-18} \text{ ues cm}$$

we get: $A_{\infty 1} = 3.5 \cdot 10^8 \text{ s}^{-1}$



Vectors $\vec{R}_{UL}$ can be arranged into a (symmetric) matrix.

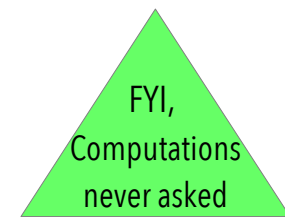
Some elements are 0 meaning that the probability of the transition is 0, namely **really prohibited**.
(for that kind of transition)

Each element $\vec{R}_{UL}$ in the matrix may be interpreted as oscillator strength for the transition $U \rightarrow L$

Selection rules for electric dipole transitions

Δn arbitrary (but not 0 for H and hydrogenoid ions)

$\Delta l = \pm 1$ (parity change) $\Delta L = 0, \pm 1, \Delta S = 0, \Delta J = 0, \pm 1$



(for reference to notation see Rohlfs, 1994 "Modern physics from α to Z_0 ")

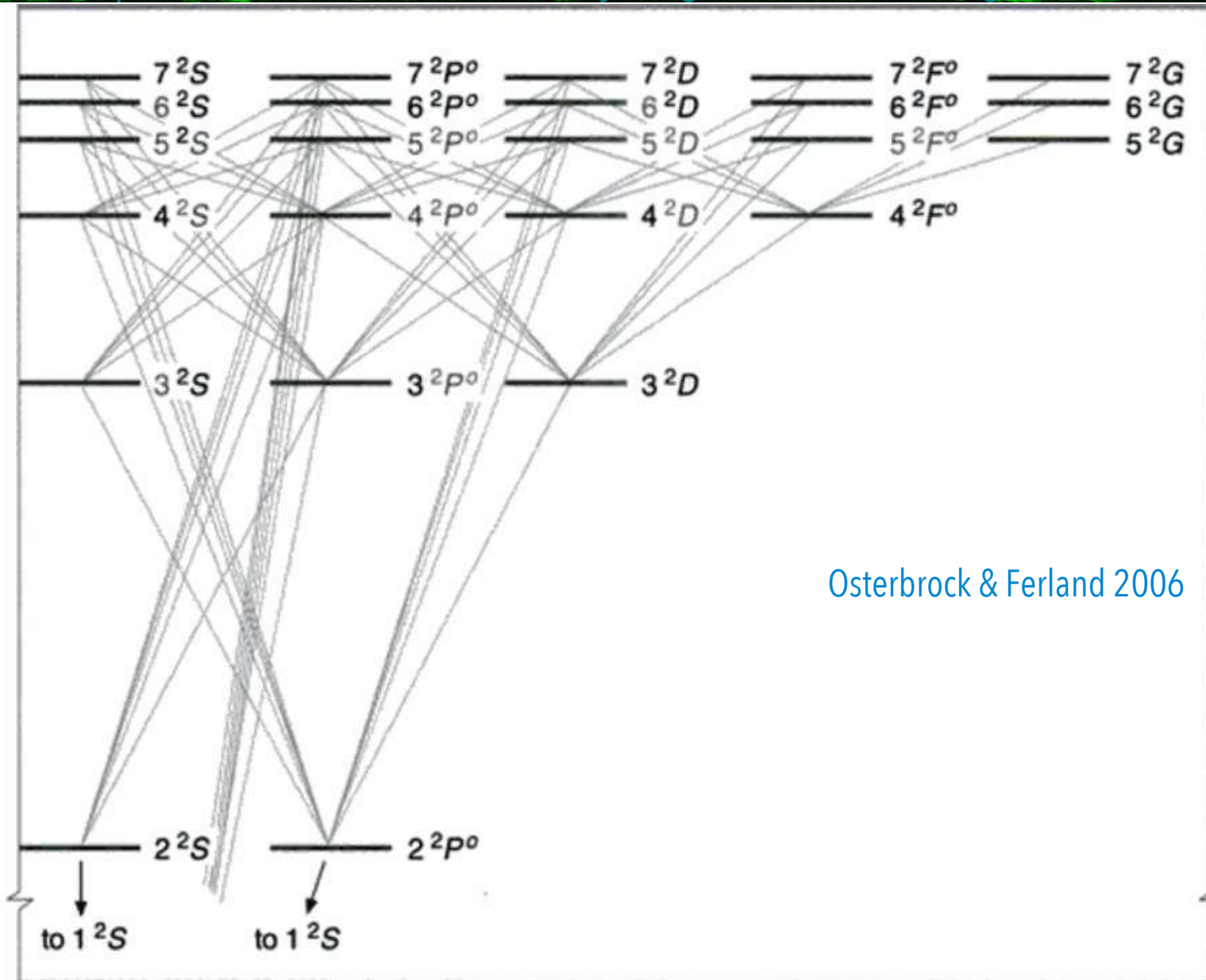


Some examples of A_{mn} coefficients for electric dipole transitions

Line	wavelength (Å)	oscillator strength	$A_{ul} (s^{-1})$
Lyman α	1215.67	0.41620	4.70E+008
Lyman β	1025.72	0.07910	5.58E+007
Lyman γ	972.54	0.02899	1.28E+007
Lyman limit	911.80		
H α	6562.80	0.64070	4.41E+007
H β	4861.32	0.11930	8.42E+006
H γ	4340.46	0.04467	2.53E+006
H δ	4101.73	0.02209	9.73E+005
H ϵ	3970.07	0.01270	4.39E+005
H limit	3646.00		
P α	18751.00	0.84210	8.99E+006
P β	12818.10	0.15060	2.20E+006
P γ	10938.10	0.05584	7.78E+005
P limit	8204.00		
B α	40512.00	1.03800	2.70E+006
B β	26252.00	0.17930	7.71E+005
B γ	21655.00	0.06549	3.04E+005
B limit	14584.00		

From Allen's *Astronomical quantities* (p 70, 71) – **warning!**

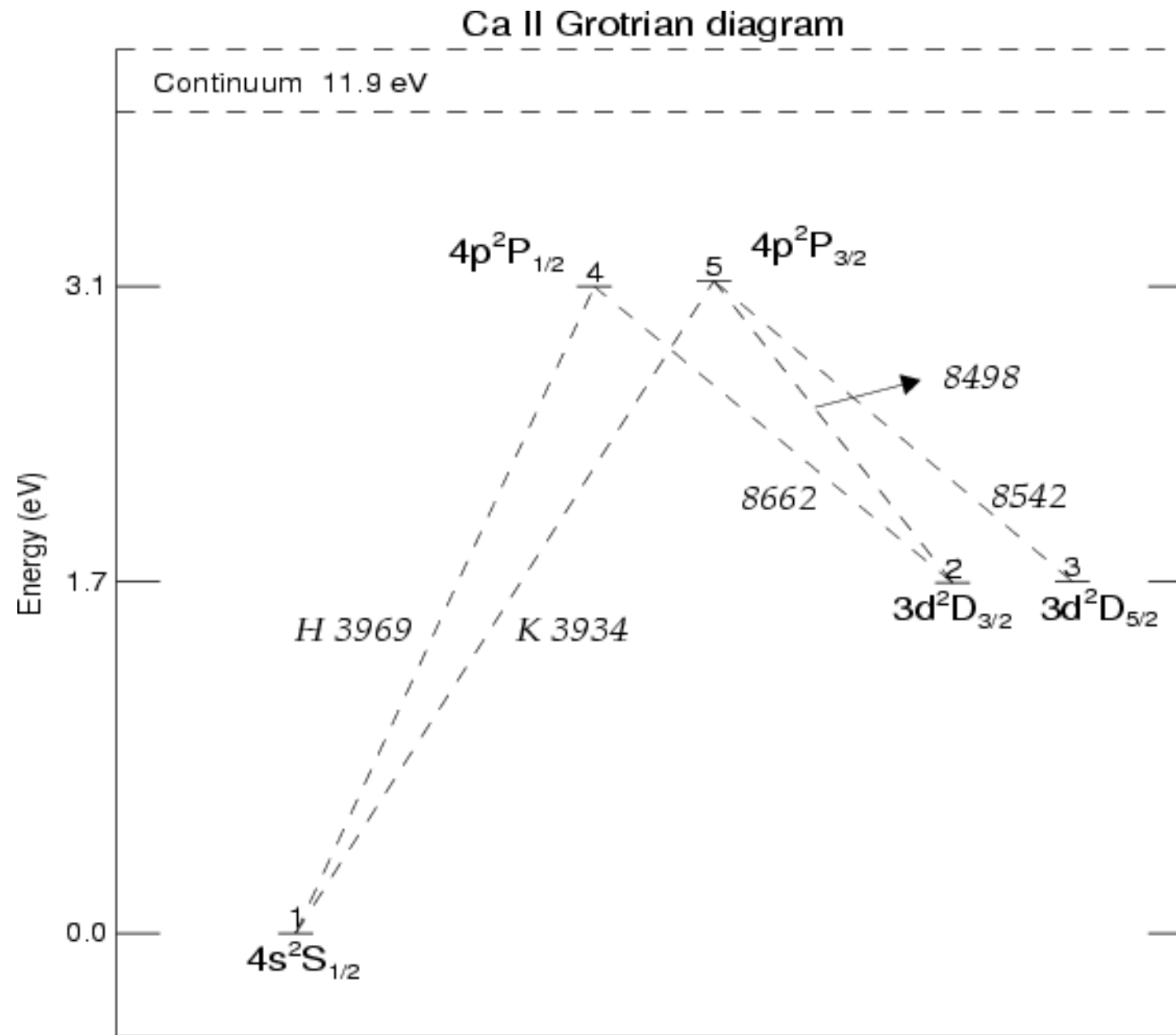
A_{ul} different from Dopita & Sutherland "Astrophysics of the Diffuse Universe", p 19



Osterbrock & Ferland 2006

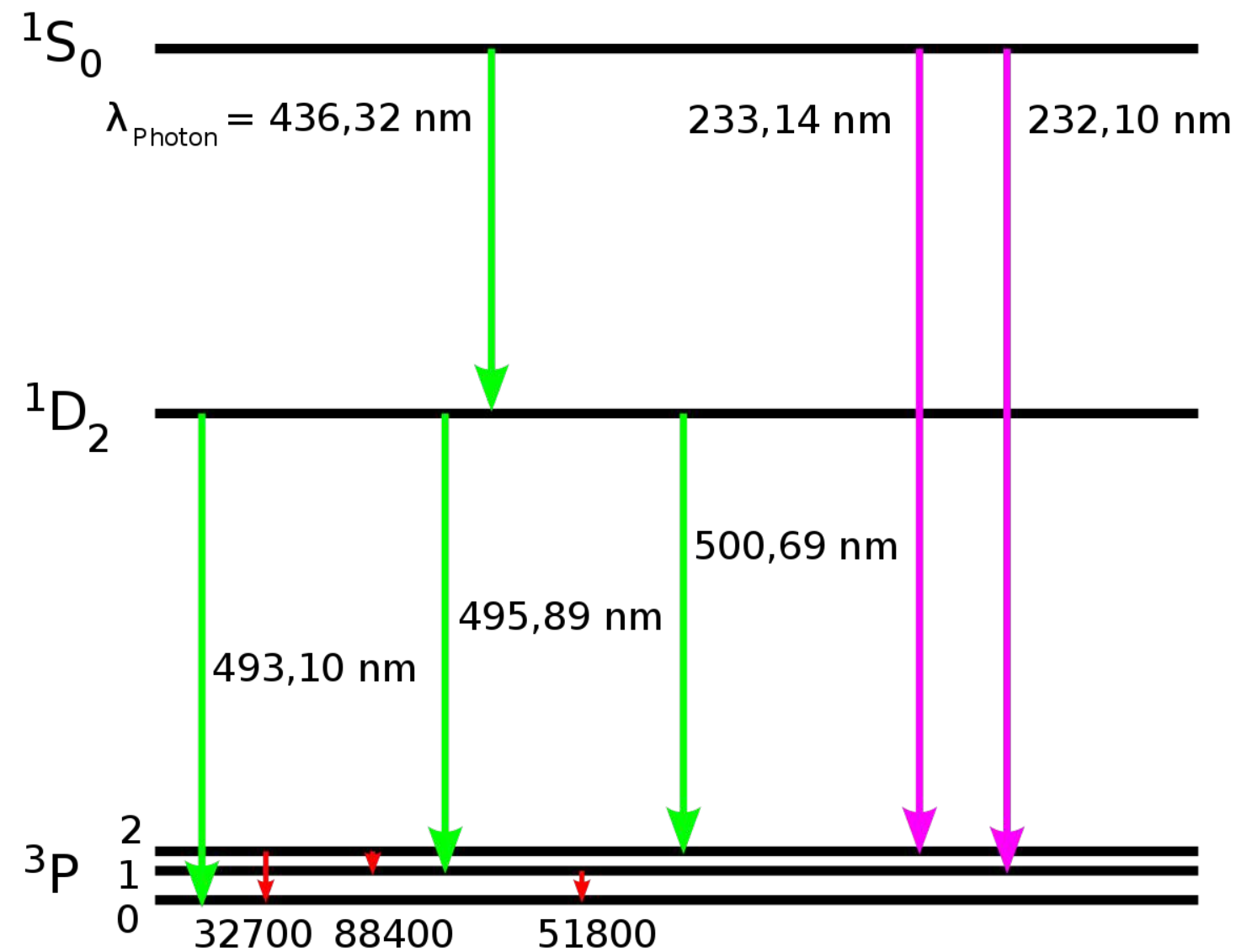
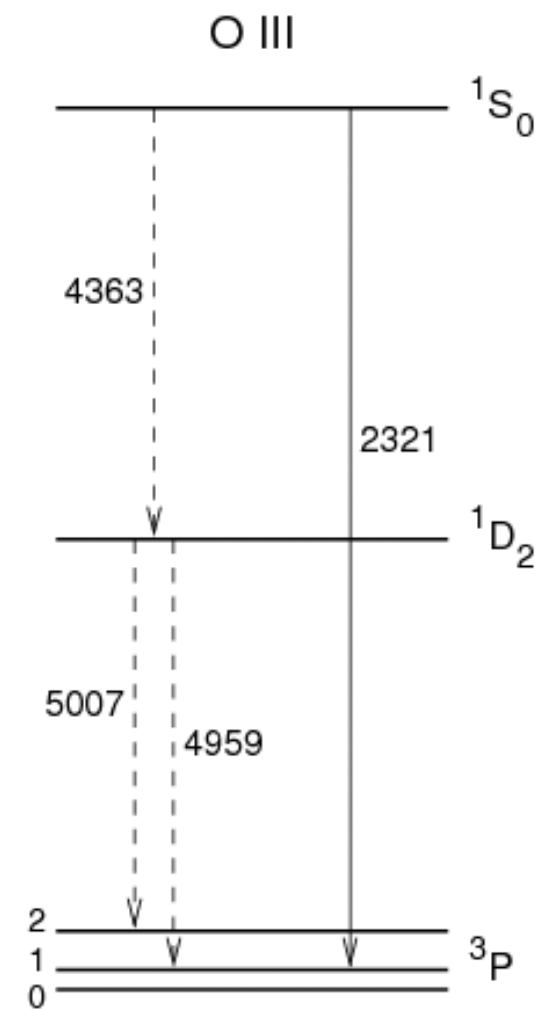
Figure 2.1

Partial energy-level diagram of H I, limited to $n \leq 7$ and $L \leq G$. Permitted radiative transitions to levels $n < 4$ are indicated by solid lines.



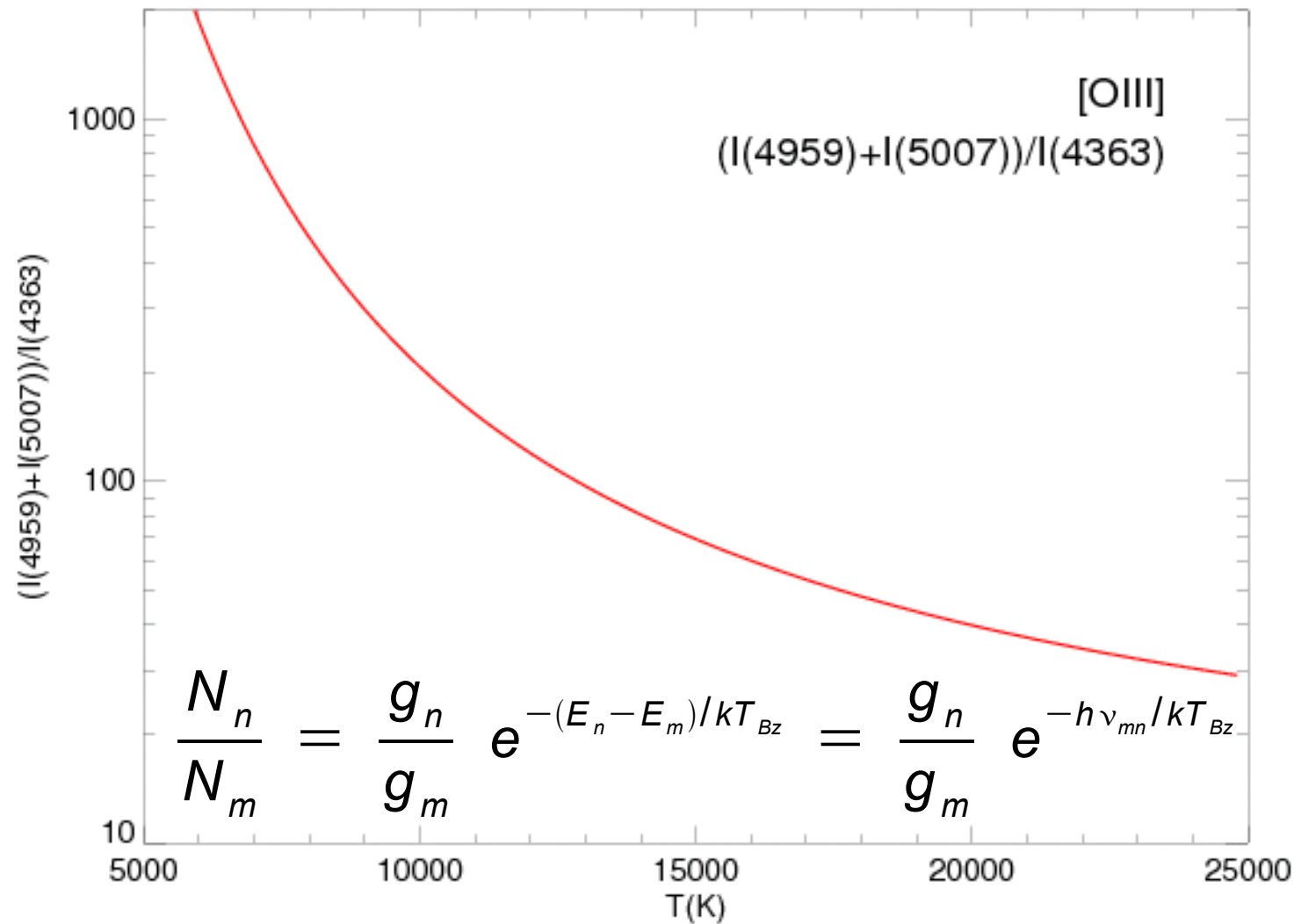
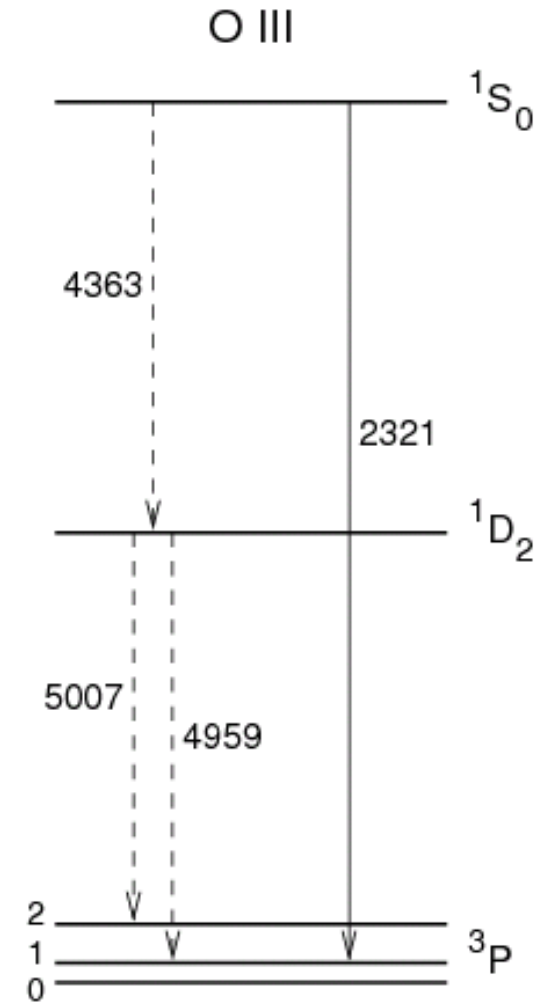


Examples: [O III] lines





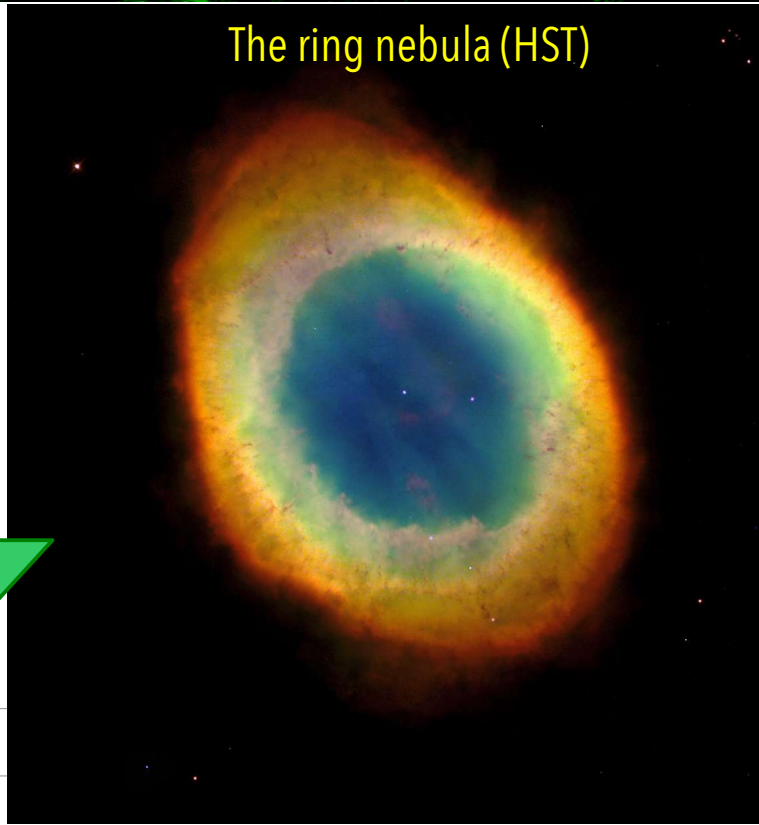
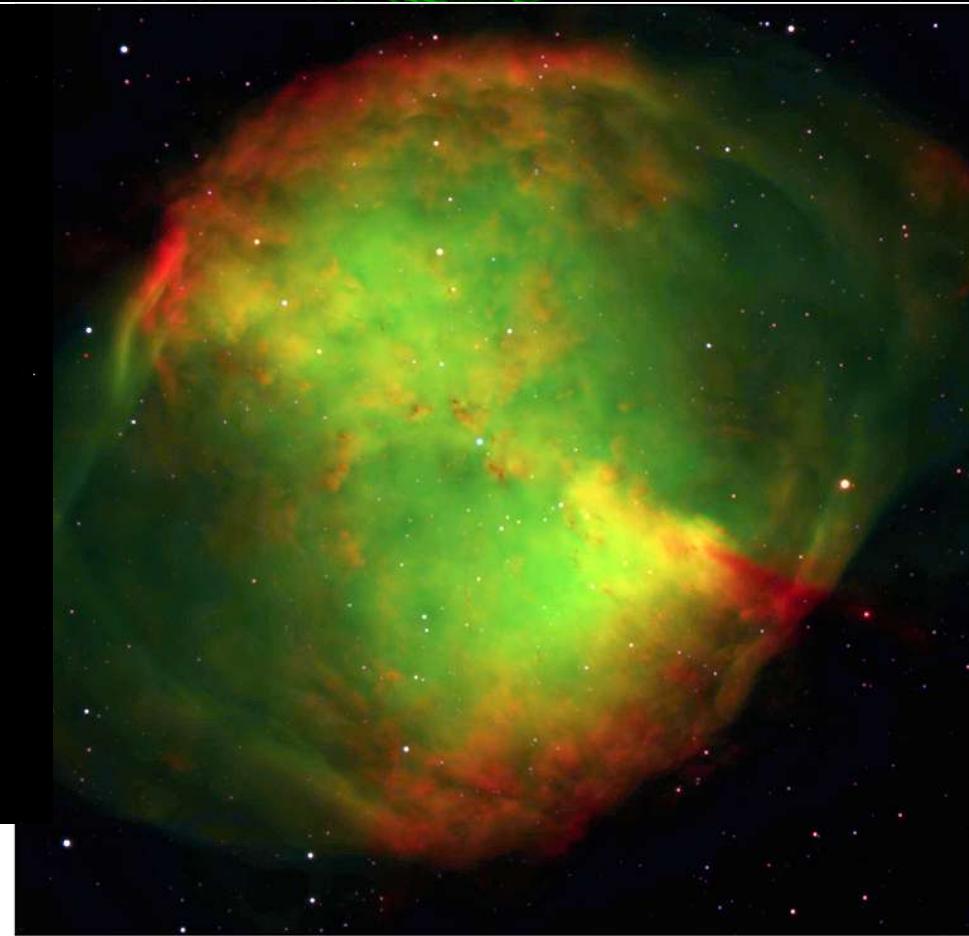
[OIII] lines diagnostics



Line ratios provide information on the temperature (note that these values are typical of HII regions!)

Forbidden lines: Nebulium (alias [OIII]) (note color stratification!)

The ring nebula (HST)

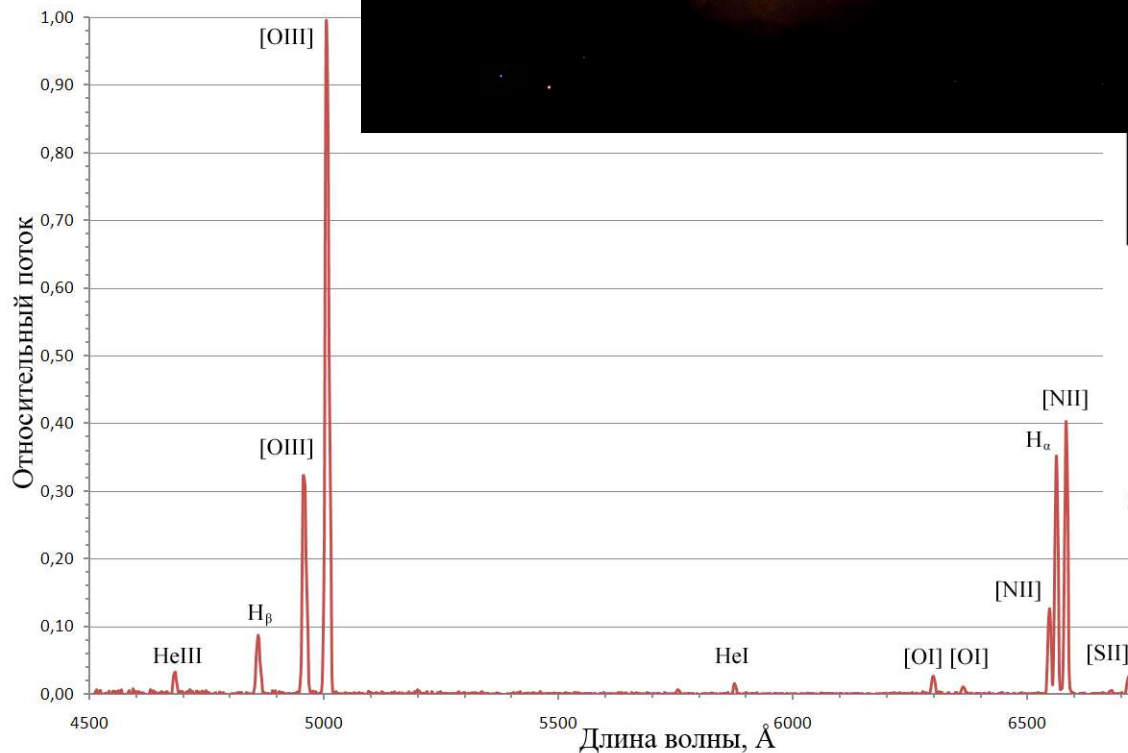


blue= broad band blue filter; green: OIII; red=H α

Planetary Nebula NGC 6853 (M 27) - VLT UT1+FORS1

ESO PR Photo 38a/98 (7 October 1998)

© ESO European Southern Observatory





Four Statistical laws (for thermal equilibrium)

Maxwell – Boltzmann (velocities .aka. **energy** for particles)

$$\frac{dN}{dv} \propto f(v) \propto v^2 e^{-mv^2/2kT_k} \approx v^2 e^{-v^2/2\sigma^2} \quad \text{where} \quad \sigma = \frac{kT_k}{m}$$

Planck (spectral distribution for radiation)

$$U(\nu) = \frac{4\pi}{c} B_\nu(T_P) = \frac{8\pi h \nu^3}{c^3} \frac{1}{e^{h\nu/kT_P} - 1}$$

Boltzmann (energy level occupation in atoms, molecules)

$$\frac{N_U}{N_L} = \frac{g_U}{g_L} e^{-(E_U - E_L)/kT_{Bz}} = \frac{g_U}{g_L} e^{-h\nu_{UL}/kT_{Bz}}$$

Saha (ionization states)

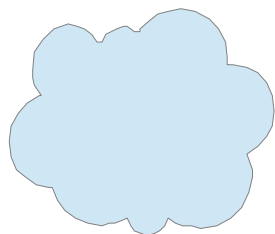
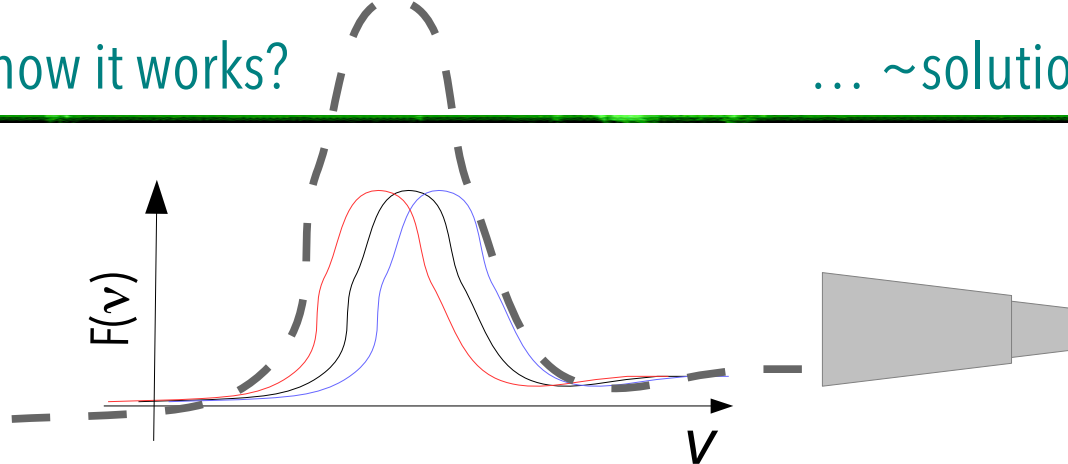
$$\frac{N_{i+1}}{N_i} = 2 \frac{u_{i+1}}{u_i} \frac{1}{N_e} \sqrt{\left(\frac{2\pi m_e kT_S}{h^2} \right)^3} e^{-(E_{i+1} - E_i)/kT_S}$$

where $\left(\sqrt{\frac{2\pi m_e kT_S}{h^2}} \right)^{-1} = \Lambda_{th}$ de Broglie thermal wavelength, and $u_i = \sum_n g_{i,n} e^{-\chi_{i,n}/kT}$

question 1:

(e.g. a standing cloud)

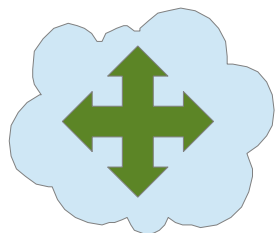
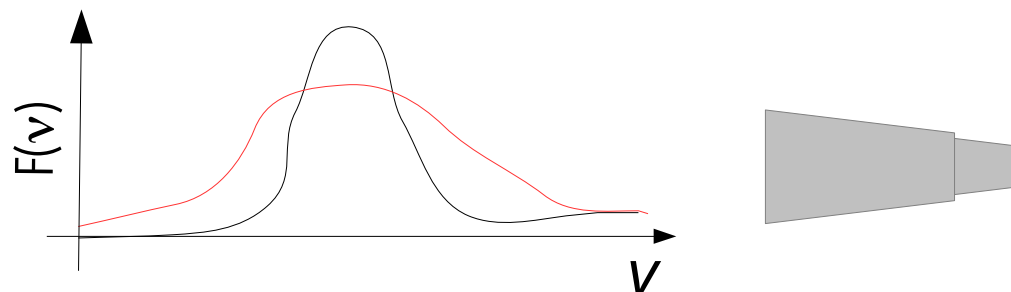
Doppler broadening from T



question 2:

(e.g. an expanding cloud)

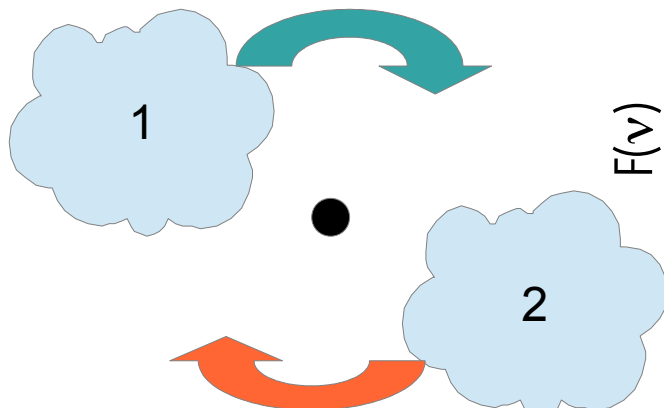
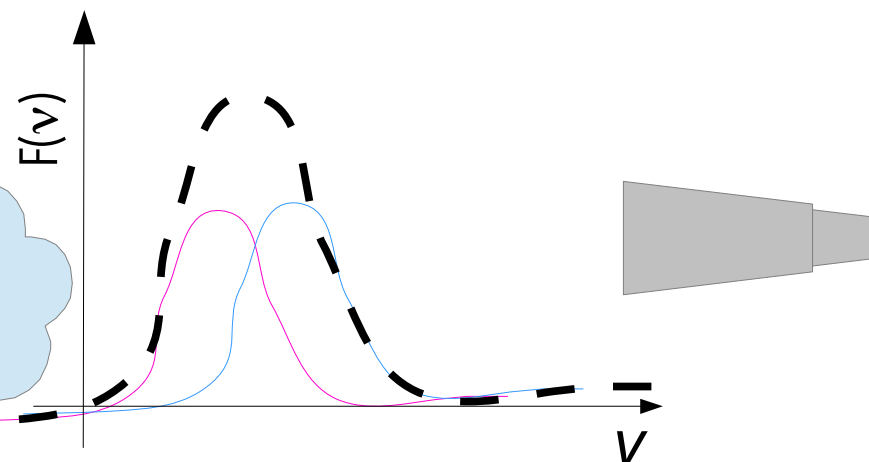
Doppler broadening from T and v



question 3:

(2 clouds orbiting around a point)

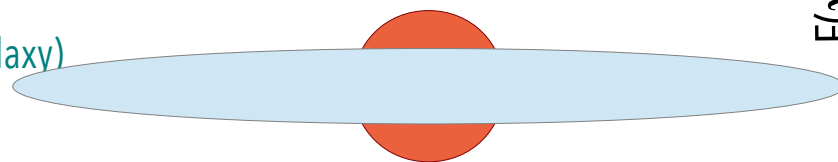
Doppler broadening from T and v_1 and v_2



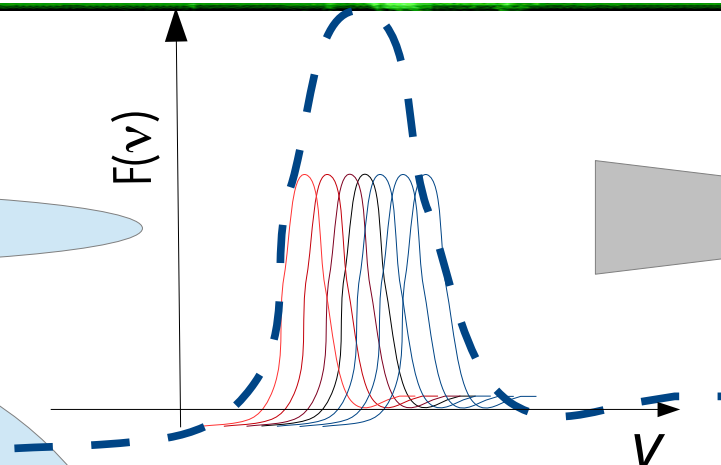
Test (2) : Do we **really** understand how it works?

question 4:

(e.g. an edge-on spiral galaxy)

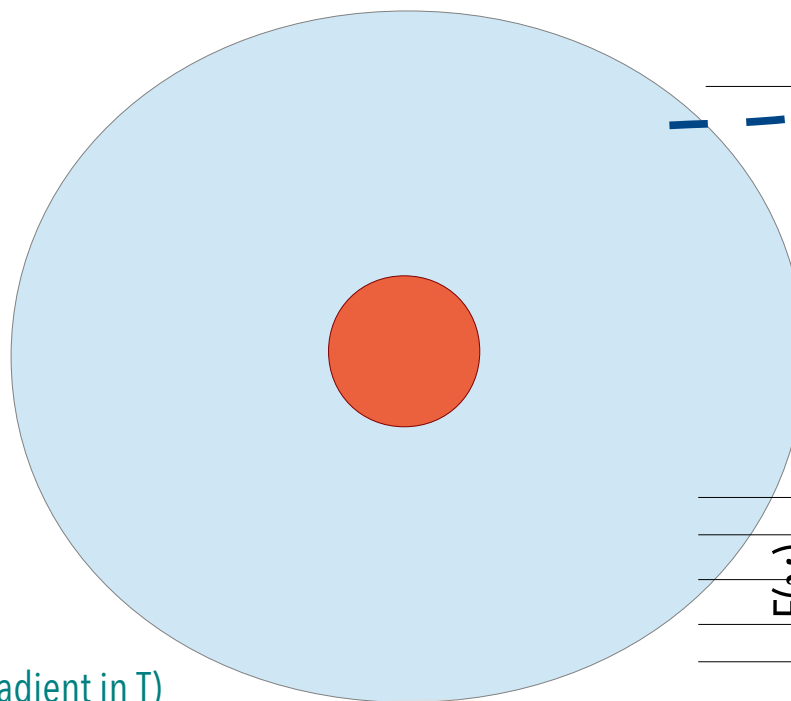


$F(v)$

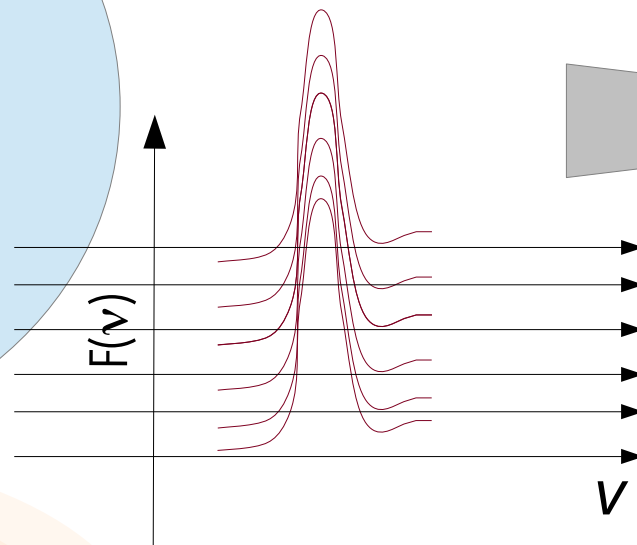


question 5:

(e.g. a face-on spiral galaxy)

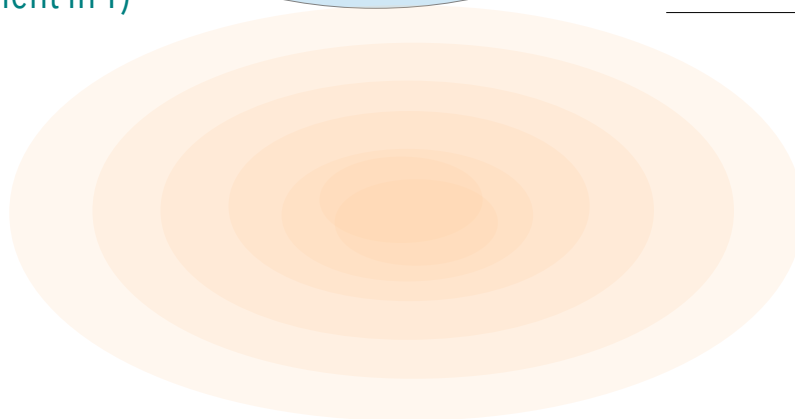


$F(v)$

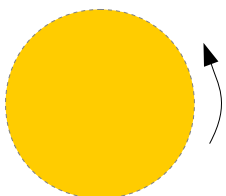


question 6:

(e.g. a stratified gas with a gradient in T)



$F(v)$



question 7:

(e.g. a rotating star)