Thermal **Bremsstrahlung**

*"Radiation due to the acceleration of a charge in the Coulomb field of another charge is called **bremsstrahlung** or **free-free** emission*

A full understanding of the process requires a quantum treatment, since photons of energies comparable to that of the emitting particle can be produced

However, a classical treatment is justified in some regimes, and the formulas so obtained have the correct functional dependence for most of the physical parameters. Therefore, we first give a classical treatment and then state the quantum results as corrections (Gaunt factors) to the classical formulas"

George B. Rybicky & Alan P. Lightman in "Radiative processes in astrophysics" p. 155

Radiative processes: foreword

Continuum from radiative processes from free-free transitions

c. physics holds when $\lambda_{dB}=\sqrt{\hbar m k T}$ *smaller that the typical scale* d *of the interaction and quantum mechanics* ($\Delta x \Delta p \geq \hbar$) *can be avoided*

$$\hbar/mv << d$$

h = Planck's constant
m = mass of the (lightest) particle
v = velocity of the (lightest) particle
d = characteristic scale (of the interaction or λ of the electromagnetic wave if it is involved)
λ_{dB} = de Broglie wavelength


Luis (7th duke of) de Broglie 1892 -1987 (Nobel 1929)

implications:
particles like points --- $h\nu << E_{particle}$

Free electrons are accelerated (decelerated) in the Coulomb field of atomic (ionized) nuclei (free-free) and radiate energy

Electrons and nuclei are in thermal equilibrium at the temperature T

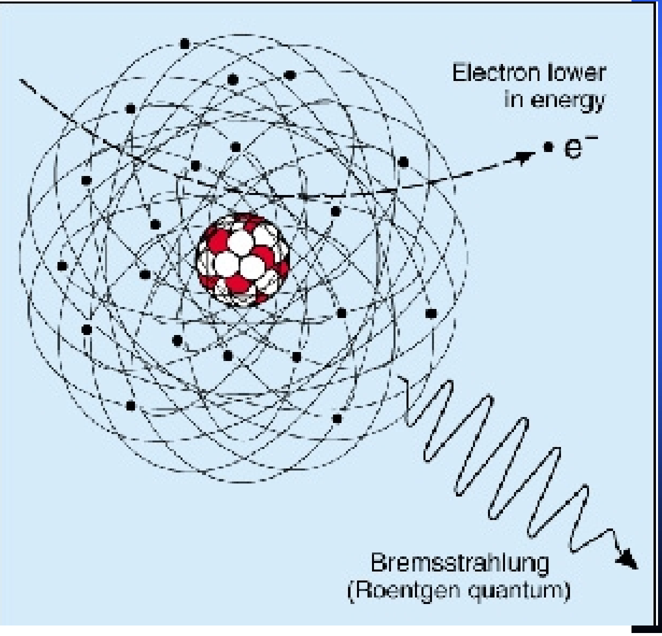
Electrons move at velocity v wrt the ions and during "collisions" are deflected. The electrostatic interaction decelerates the electron which emits a photon bringing away part of its kinetic energy

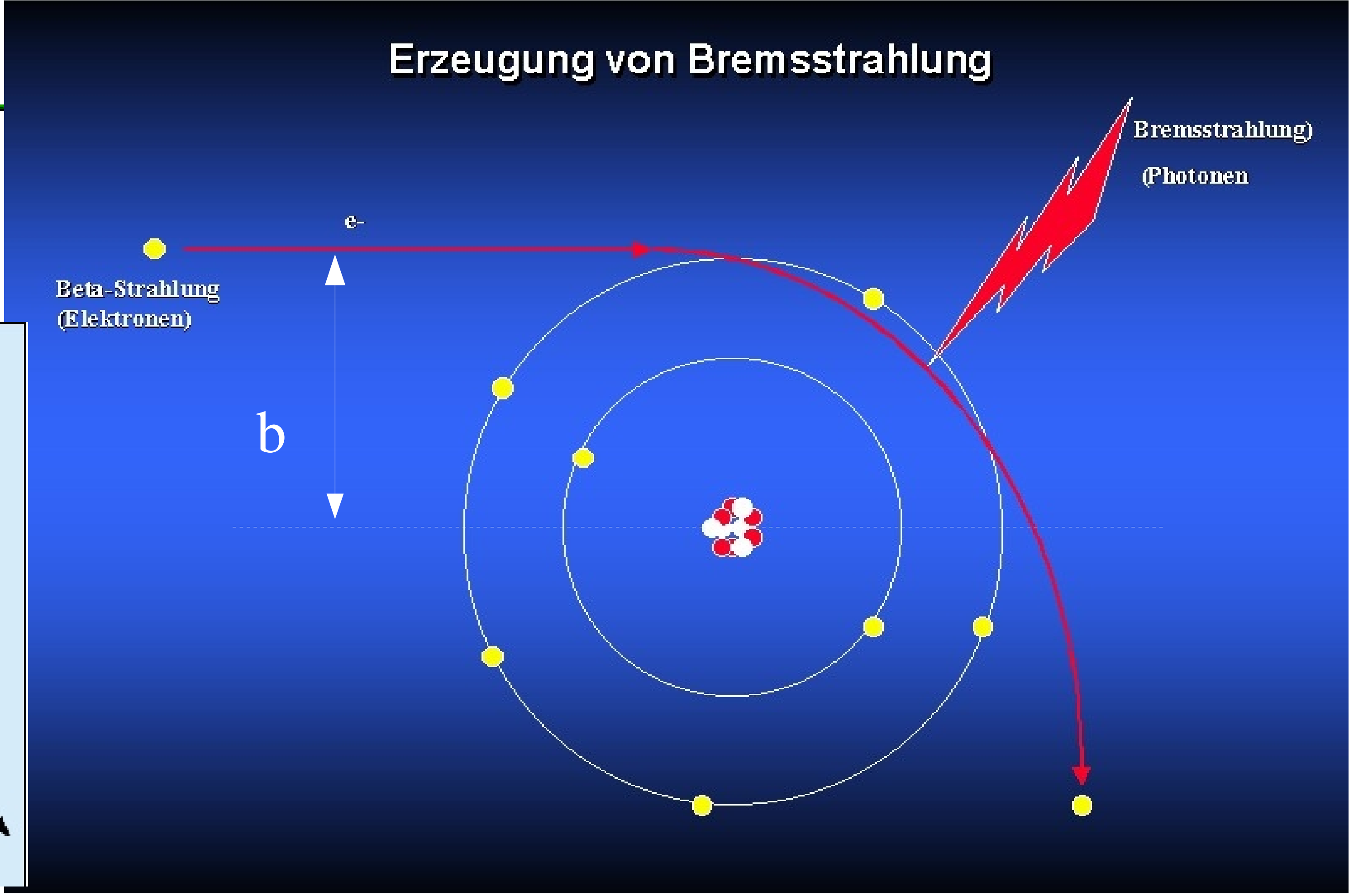
*N.B. **e** and **p** have also homologous interactions (**e** with **e**; **p** with **p**) but they do not emit radiation since the electric dipole is zero.*

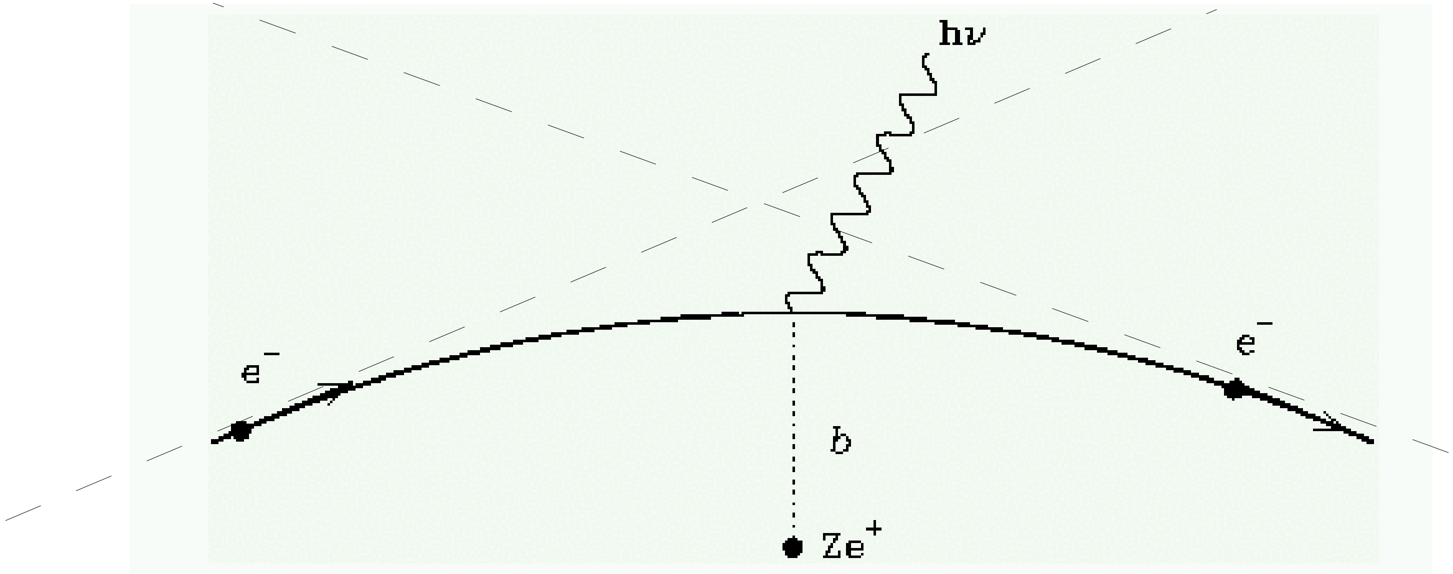
Astrophysical examples:

1. *HII regions (10^4 °K)*
2. *Galactic hot-coronae (10^7 °K)*
3. *Intergalactic gas in clusters/groups (10^{7-8} °K)*
4. *Accretion disks (around evolved stars & BH)*

Bremsstrahlung is the main cooling process for high T (> 10^7 K) plasma

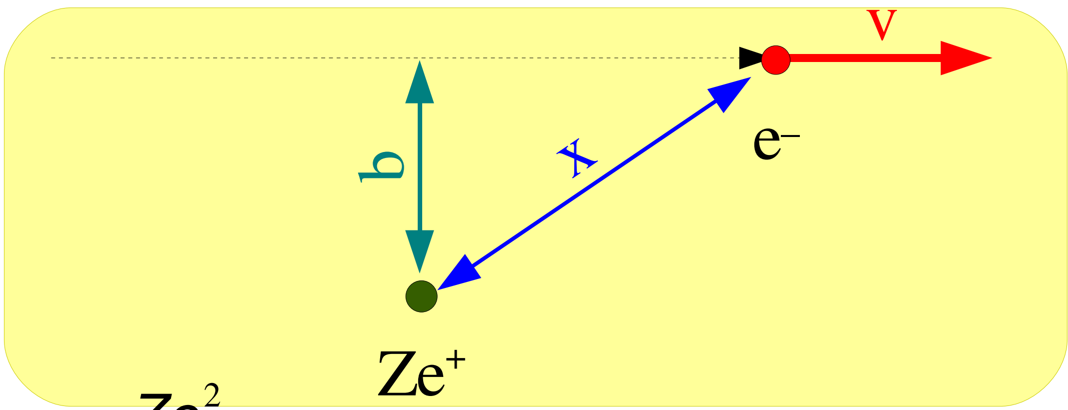

Electron lower
in energy
e⁻
Bremsstrahlung
(Roentgen quantum)


Erzeugung von Bremsstrahlung
Bremsstrahlung)
(Photonen
e-
Beta-Strahlung
(Elektronen)
b


hν
e⁻
e⁻
b
Ze⁺

Radiated power (single particle) is given by the Larmor's formula:

$$P = -\frac{dE}{dt} = \frac{2}{3}\frac{e^2}{c^3}\, a^2$$



acceleration is from Coulomb's law

$$F = ma \simeq -\frac{Ze^2}{x^2} \quad \rightarrow \quad a = \frac{Ze^2}{m x^2}$$

where x= actual distance between e^- and Ze^+

Remarks:

1. $a \sim m^{-1} \rightarrow P \sim m^{-2}$
2. $a \sim x^{-2} \rightarrow P \sim x^{-4}$

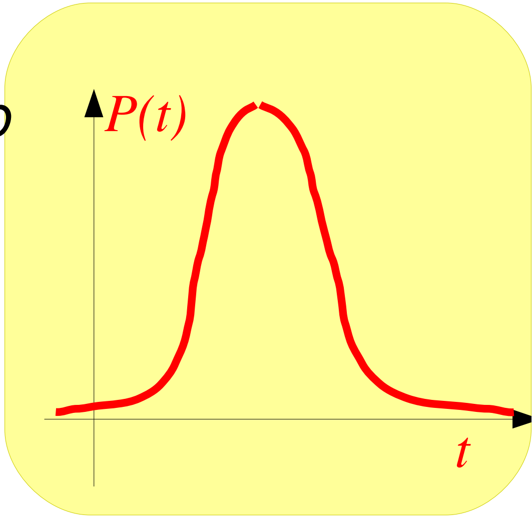
Radiated power is relevant

1. *for electrons (positrons) only i.e. light particles*
2. *when the two charges are at the closest distance ($x \sim b$)*

The interaction lasts very shortly: $\Delta t \sim 2b/v$ *where* v *= velocity*

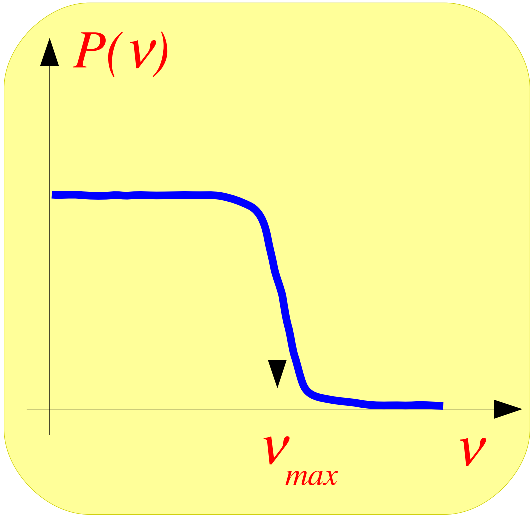
The total emitted energy in a single "collision" (consider $x \sim b$)

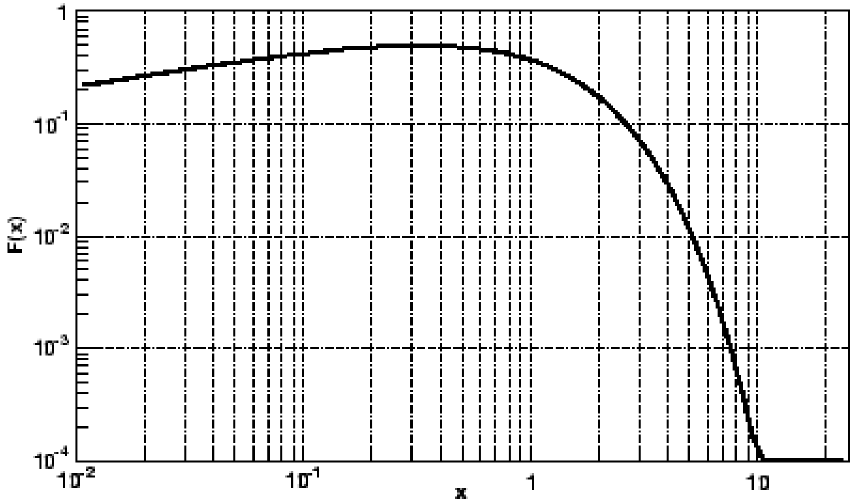
$$P\,\Delta t = \frac{2}{3}\frac{e^2}{c^3}\left(\frac{Z e^2}{m x^2}\right)^2 \frac{2b}{v} = \frac{4}{3}\frac{Z^2 e^6}{c^3 m^2}\frac{1}{b^3 v} \qquad x \simeq b$$



Radiation comes in pulses Δt long: the profile is slightly asymmetric due to the decrease of the e^- velocity

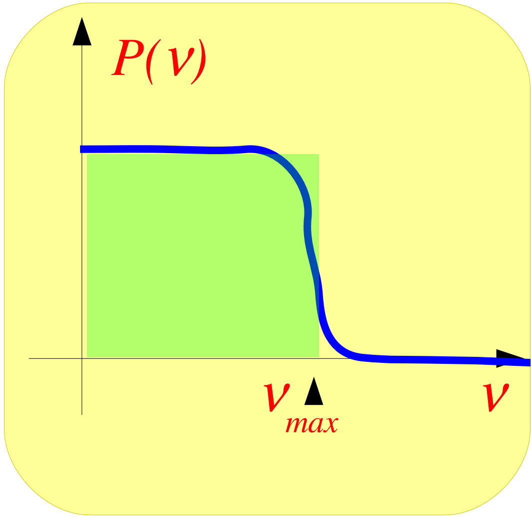
The Fourier analysis of the pulse provides the spectral distribution of the radiated energy: it is flat up to a frequency





$$\nu_{max} \simeq \frac{1}{2\,\Delta t} \approx \frac{v}{4b}$$

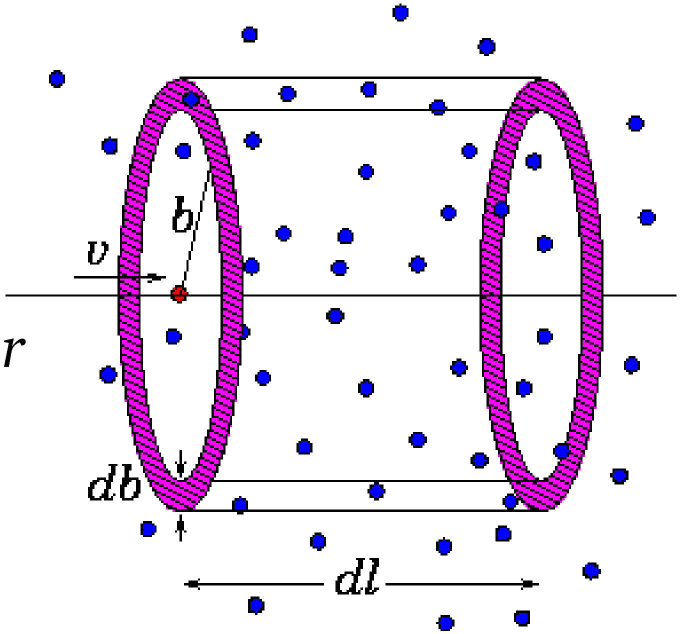
related to the kinetic energy of the electrons (exponential decay beyond ν_{max})



The radiated energy per unit frequency (assuming flat spectrum) from a single event can be written as

$$\frac{P\,\Delta t}{\Delta \nu} \approx \frac{P\,\Delta t}{\nu_{max}} = 2P(\Delta t)^2 \simeq \frac{16}{3}\frac{Z^2 e^6}{c^3 m_e^2}\frac{1}{b^2 v^2} \approx \frac{1}{b^2 v^2}$$

The real case: a cloud with ion number density n_Z and free electron number density n_e



In a unit time, each electron experiences a number of collisions with an impact parameter between b and $b + db$ is

$$2\pi n_Z v\, b\, db$$

and the total number of collisions is

$$2\pi n_e n_Z v\, b\, db$$

Total emissivity from a cloud emitting via bremsstrahlung (single velocity!)

$$J_{br}(v,\nu) = \frac{dW}{d\nu\, dV\, dt} = 2\pi n_e n_Z v \int_{b_{min}}^{b_{max}} \frac{16}{3}\frac{Z^2 e^6}{c^3 m_e^2}\frac{1}{b^2 v^2}\, b\, db =$$

$$= \frac{32\pi e^6}{3 c^3 m_e^2 v} n_e n_Z Z^2 \int_{b_{min}}^{b_{max}} \frac{db}{b} = \frac{32\pi e^6}{3 c^3 m_e^2}\frac{1}{v} n_e n_Z Z^2 \ln\left(\frac{b_{max}}{b_{min}}\right)$$

Evaluation of b_{max}:

at a given ν, we must consider only those interactions where electrons have impact parameters corresponding to $\nu_{max} > \nu$ and then

$$b_{max} \leq \frac{v}{4\nu}$$

is a function of frequency

Evaluation of b_{min}:

a) classical physics requires that $\Delta v \leq v$

$$\Delta v \simeq a \Delta t = \frac{Ze^2}{m_e b^2} \frac{2b}{v} \leq v \quad \rightarrow \quad b_{min-c} \geq \frac{2Ze^2}{m_e v^2}$$

b) quantum physics and the Heisemberg principle $\Delta p \Delta x \geq h/2\pi$

$$\Delta p = m_e \Delta v \approx m_e v \geq \frac{h}{2\pi \Delta x} \approx \frac{h}{2\pi b_{min}} \quad \rightarrow \quad b_{min-q} \geq \frac{h}{2\pi m_e v}$$

⇒ we must consider which b_{min} has to be used *which [classical/quantum] limit prevails:*

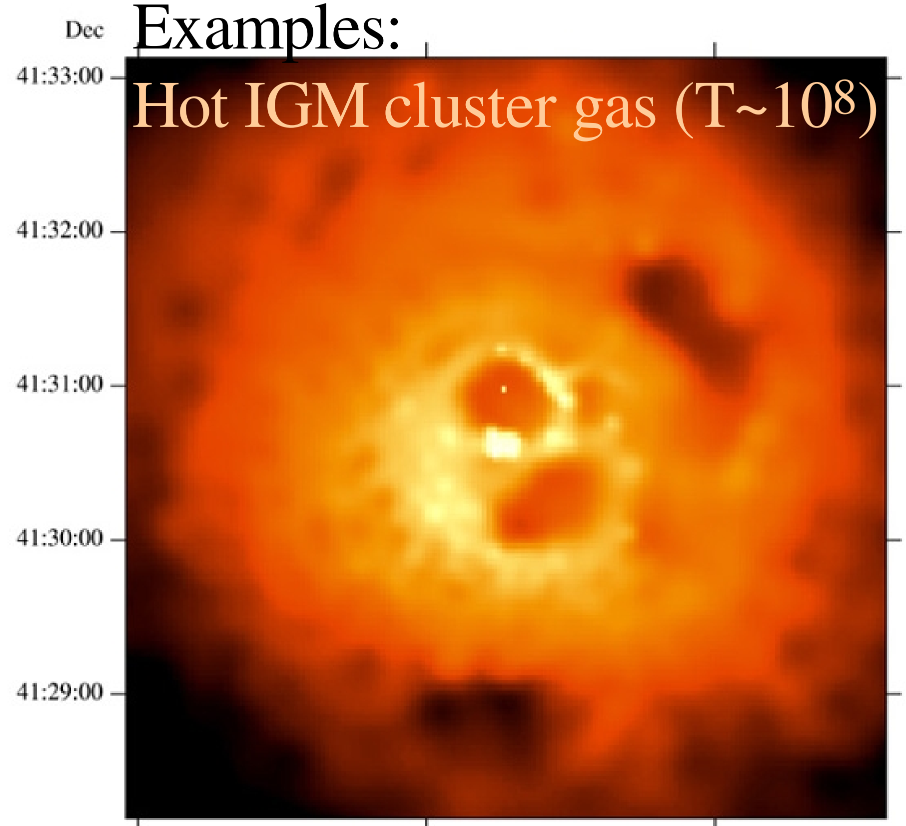
$$\frac{b_{min-q}}{b_{min-c}} \approx \frac{h/(2\pi m_e v)}{2Ze^2/(m_e v^2)} = \frac{h}{4\pi Z e^2} c \frac{v}{c} \approx \frac{137}{Z}\frac{v}{c}$$

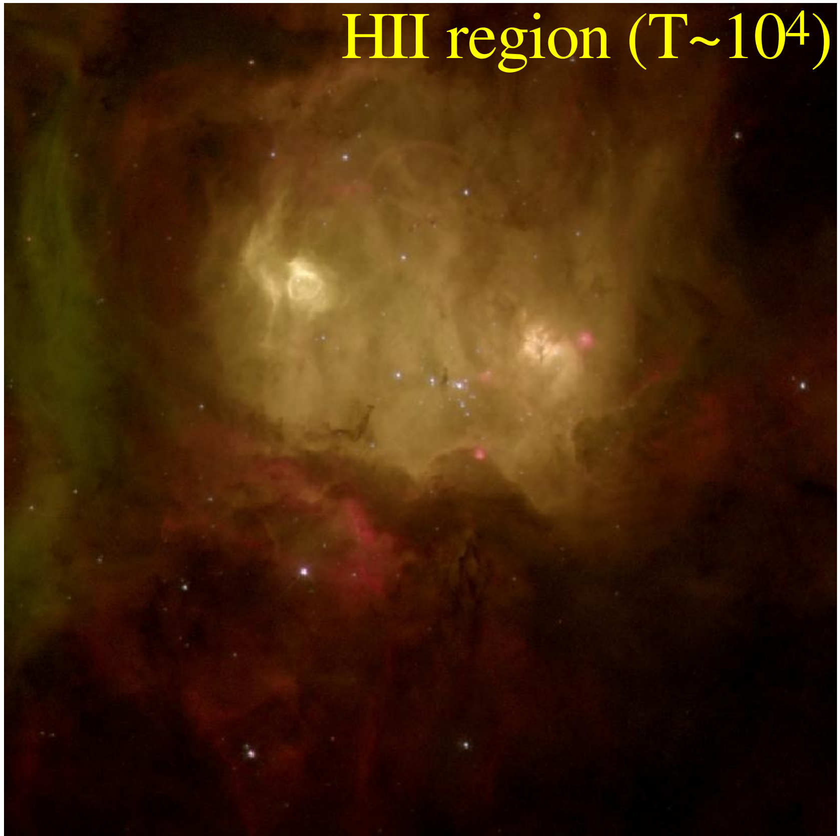
≥1 when $v \geq 0.01c$

but

$$v = \sqrt{3kT/m_e} \quad \text{then} \quad \frac{b_{min-q}}{b_{min-c}} \approx \frac{137}{Zc}\sqrt{\frac{3k}{m_e}}\,T^{1/2}$$

Examples:



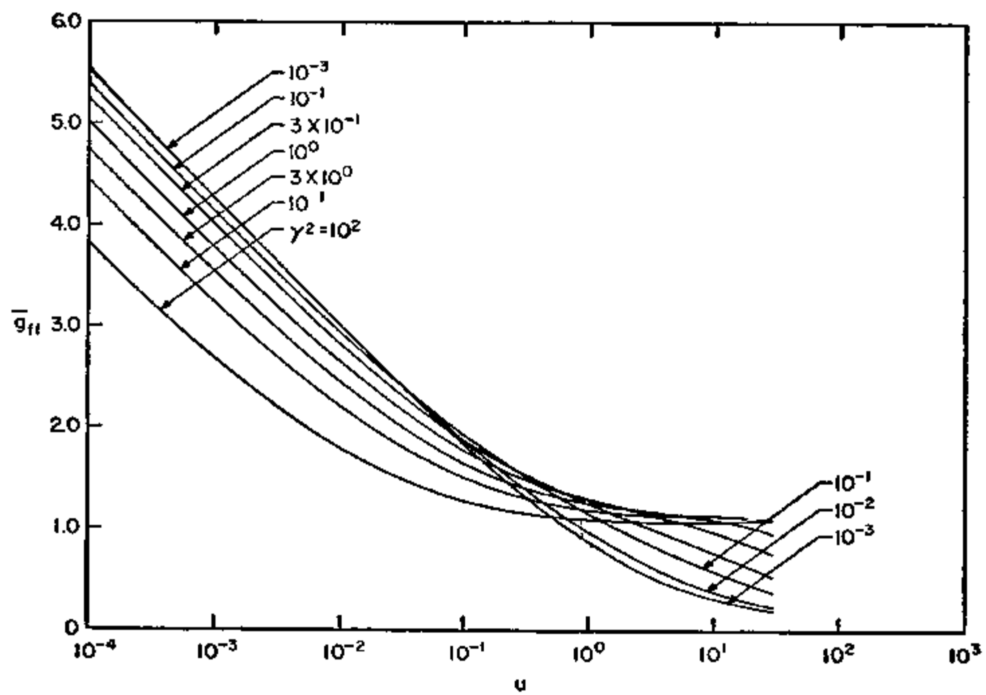


$$g_{ff}(\nu, T) = \frac{\sqrt{3}}{\pi} \ln(\frac{b_{max}}{b_{min}})$$

The Gaunt Factor ranges from about 1 to about 10 in the radio domain

5–11

Thermal Bremsstrahlung



Gaunt factor, $\bar{g}_{\rm ff}$, from Karzas & Latter, Fig. 5. $u = h\nu/kT$, for various values of $\gamma^2 = Z^2\,{\rm Ry}/kT$, where 1 Ry = 13.6 eV.

IAAT

Values around 10-15 in the radio domain, slightly higher than 1 at higher energies

A set of particles at thermal equilibrium follow Maxwell-Boltzmann distribution:

$$f(v)dv = 4\pi\left(\frac{m_e}{2\pi k T}\right)^{3/2} e^{-m_e v^2/2kT} v^2 dv$$

Then $n_e(v) = n_e f(v)\, dv$ *is the number density of electrons whose velocity is between* v *and* $v+dv$ *and* $n_e(v)$ *replaces* n_e *in* $J_{br}(\nu)$

When the plasma is at equilibrium, the process is termed

thermal bremsstrahlung

Total emissivity:

must integrate over velocity (energy) distribution ($0 \le v < \infty$ ***?****)*

$$J_{br}(\nu, T) = \frac{dW}{dt\, dV\, d\nu} = \int_{v_{min}}^{\infty} J_{br}(\nu, v) f(v) dv =$$

$$= 6.8 \cdot 10^{-38}\, T^{-1/2}\, e^{-h\nu/kT}\, n_e n_Z Z^2 g_{ff}(\nu, T) \quad erg\, s^{-1} cm^{-3} Hz^{-1}$$

T ~ particle energy

high energy cut-off (T)

bremsstrahlung – interlude: photon frequency .vs. temperature

Consider an electron moving at a speed of $v = 1000 \text{ km s}^{-1}$.

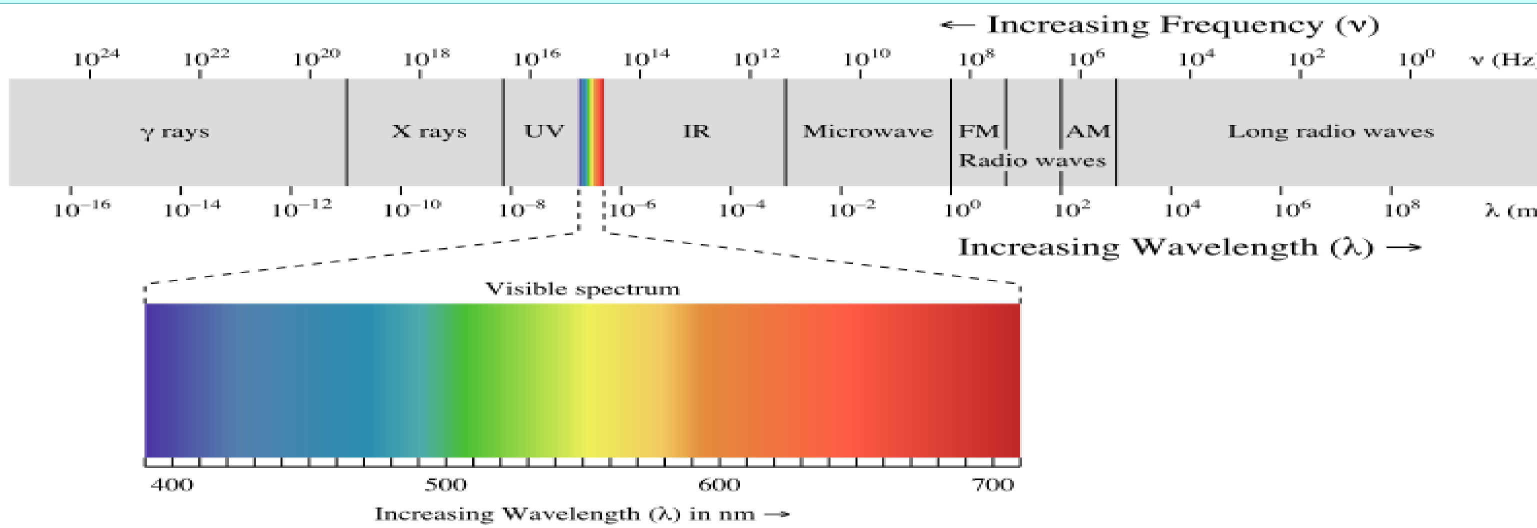
In case it would radiate all its kinetic energy in a single interaction

$$\Delta E = h\nu = \frac{1}{2} m v^2 = 0.5 \times 9.1 \times 10^{-31} \text{kg} \times (10^6 \text{m s}^{-1})^2 = 4.55 \times 10^{-19} \text{J}$$

the frequency of the emitted radiation is $\nu = \frac{\Delta E}{h} = \frac{4.55 \times 10^{-19} \text{kg m}^2 \text{s}^{-2}}{6.626 \times 10^{-34} \text{kg m}^2 \text{s}^{-1}} = 6.87 \times 10^{14} \text{Hz}$

in case this electron belongs to a population of particles at thermal equilibrium for which $v = 1000 \text{km s}^{-1}$ *is the typical speed, determine which is the temperature of the plasma*

$$T = \frac{\Delta E}{k} = \frac{4.55 \times 10^{-19} \text{kg m}^2 \text{s}^{-2}}{1.38 \times 10^{-23} \text{kg m}^2 \text{s}^{-2} \text{K}^{-1}} = 3.30 \times 10^4 \text{K}$$



k $= 1.3806503 \times 10^{-23} \text{ m}^2 \text{ kg s}^{-2} \text{ K}^{-1}$ *h* $= 6.626068 \times 10^{-34} \text{ m}^2 \text{ kg s}^{-1}$

$m_e = 9.10938188 \times 10^{-31}$ kg

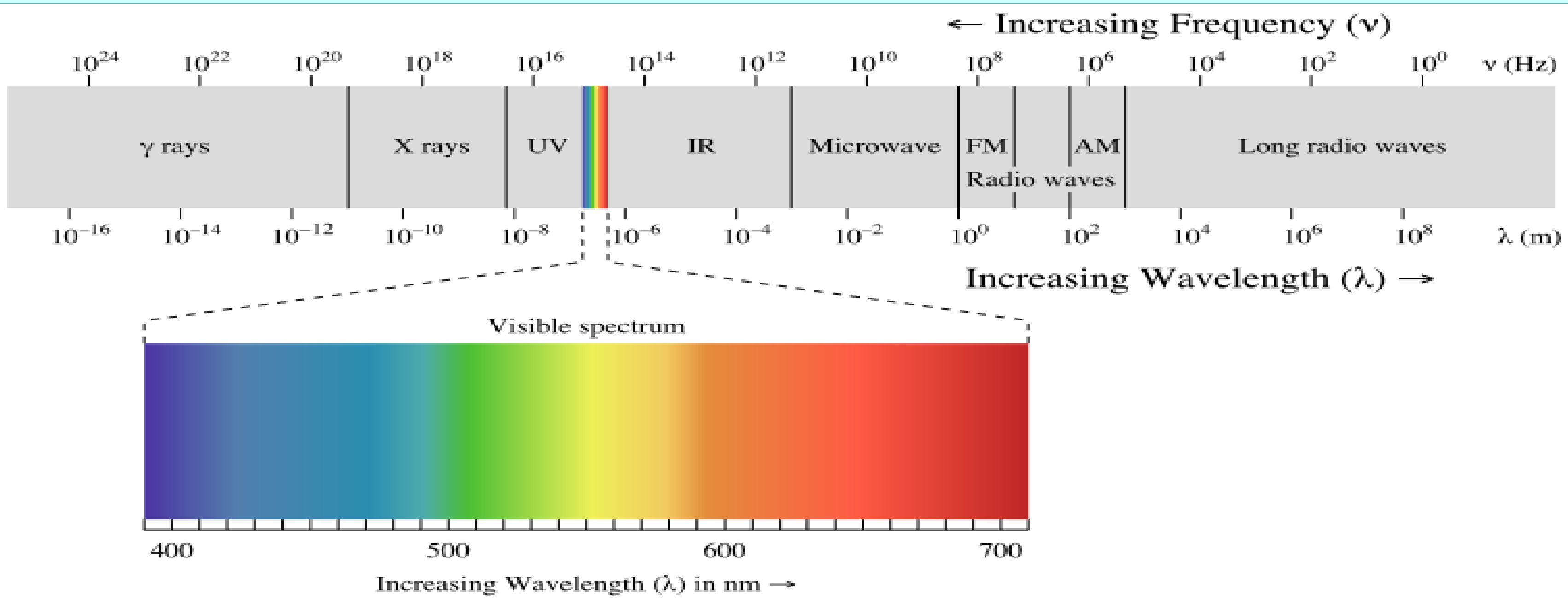
bremsstrahlung – interlude (2): the cut-off frequency

The cut-off frequency in the thermal bremsstrahlung spectrum depends on the temperature only, and is conventionally set when the exponential equals 1/e.

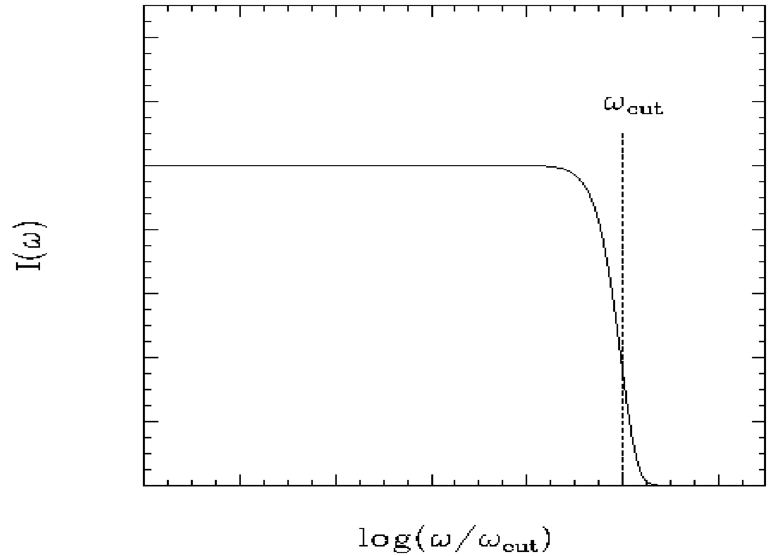
$$\frac{h\nu}{kT} = 1 \quad \rightarrow \quad \nu = \frac{k}{h}T = \frac{1.38 \times 10^{-23}\,kg\,m^2\,s^{-2}\,K^{-1}}{6.626 \times 10^{-34}\,kg\,m^2\,s^{-1}} \times T = 2.08 \times 10^{10} \times T \quad (Hz)$$

It is interesting to determine the cut-off frequency for a warm plasma (HII region, $T \sim 10^4$ K) and for a hot plasma (IGM in clusters of galaxies, $T \sim 10^8$ K).

⇨ *In observations, the measure of the cut-off frequency is a way to determine the plasma temperature*



Total emissivity does not depend on frequency (except the cut – off) and the spectrum is flat



Total emitted energy per unit volume: must integrate over velocity (energy) distribution ($0 \leq v < \infty$?)

at a given frequency ν the velocity of the particle v must be as such

$$\frac{1}{2} m v^2 \geq h\nu \qquad \rightarrow \qquad v_{min} \geq \sqrt{\frac{2h\nu}{m_e}}$$

and integrating over the whole spectrum:

$$J_{br}(T) = \frac{dW}{dt\, dV} \simeq 2.4 \cdot 10^{-27} T^{1/2} n_e n_Z Z^2 \bar{g}_{ff}(T) \qquad erg\, s^{-1} cm^{-3}$$

$\bar{g}_{ff}(T)$ *average gaunt factor*

Let's consider the total thermal energy of a plasma and the radiative losses: the **cooling time** *is*

$$t_{br} = \frac{E_{th}^{tot}}{J_{br}(T)} = \frac{3/2(n_e+n_p)kT}{J_{br}(T)}$$

we can assume $n_e = n_p$

$$\simeq \frac{3n_e kT}{2.4\cdot 10^{-27}\, T^{1/2} n_e^2 \bar{g}_{ff}} = \frac{1.8\cdot 10^{11}}{n_e \bar{g}_{ff}} T^{1/2} \quad sec \quad \rightarrow \quad = \frac{6\cdot 10^{3}}{n_e \bar{g}_{ff}} T^{1/2} \quad yr$$

N.B. The "average" Gaunt Factor is about a factor of a few

Astrophysical examples:

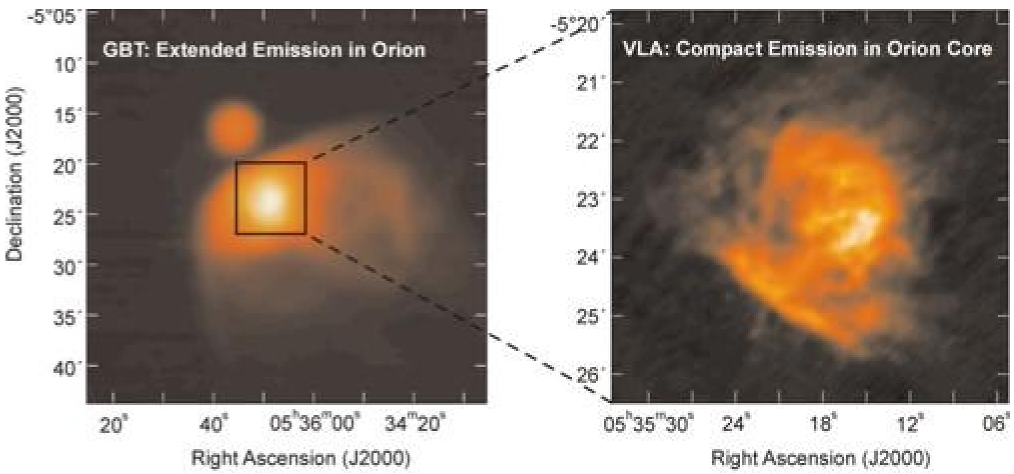
HII regions: $n_e \sim 10^2 - 10^3 \; cm^{-3}$; $T \sim 10^4 \; K$

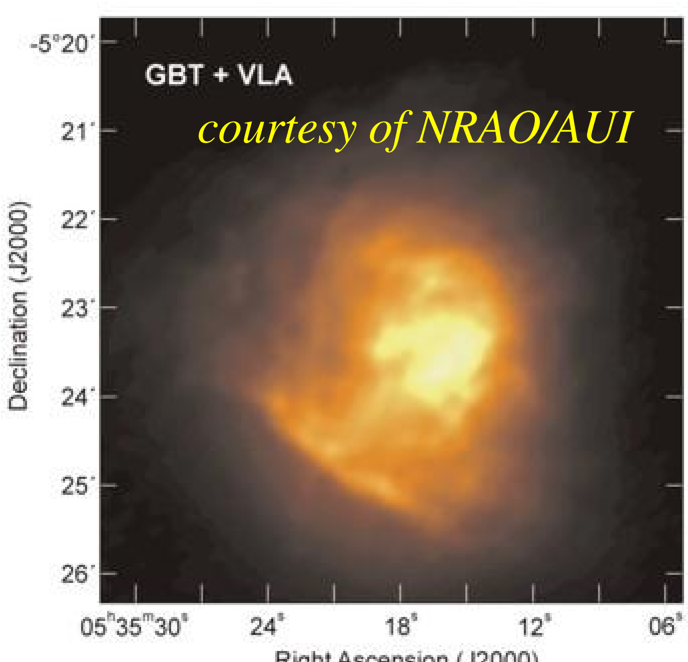
Galaxy clusters $n_e \sim 10^{-3} \; cm^{-3}$; $T \sim 10^7 - 10^8 \; K$

(thermal) bremsstrahlung is the main cooling process at temperatures above $T \sim 10^7$ K

N.B.-2- all galaxy clusters have bremsstrahlung emission

A galactic HII region: M42 (Orion nebula)
distance : 1,6 kly ~ 0.5 kpc
angular size ~ 66 ' ~ 1.1°





compute:

$$h\nu_{cutoff} \sim kT = (1.38\cdot10^{-16}\, erg\, K^{-1} \times 6.24\cdot10^{-11}\, erg/eV)\,T$$

$$= 8.61\cdot10^{-5}\,T \quad eV$$

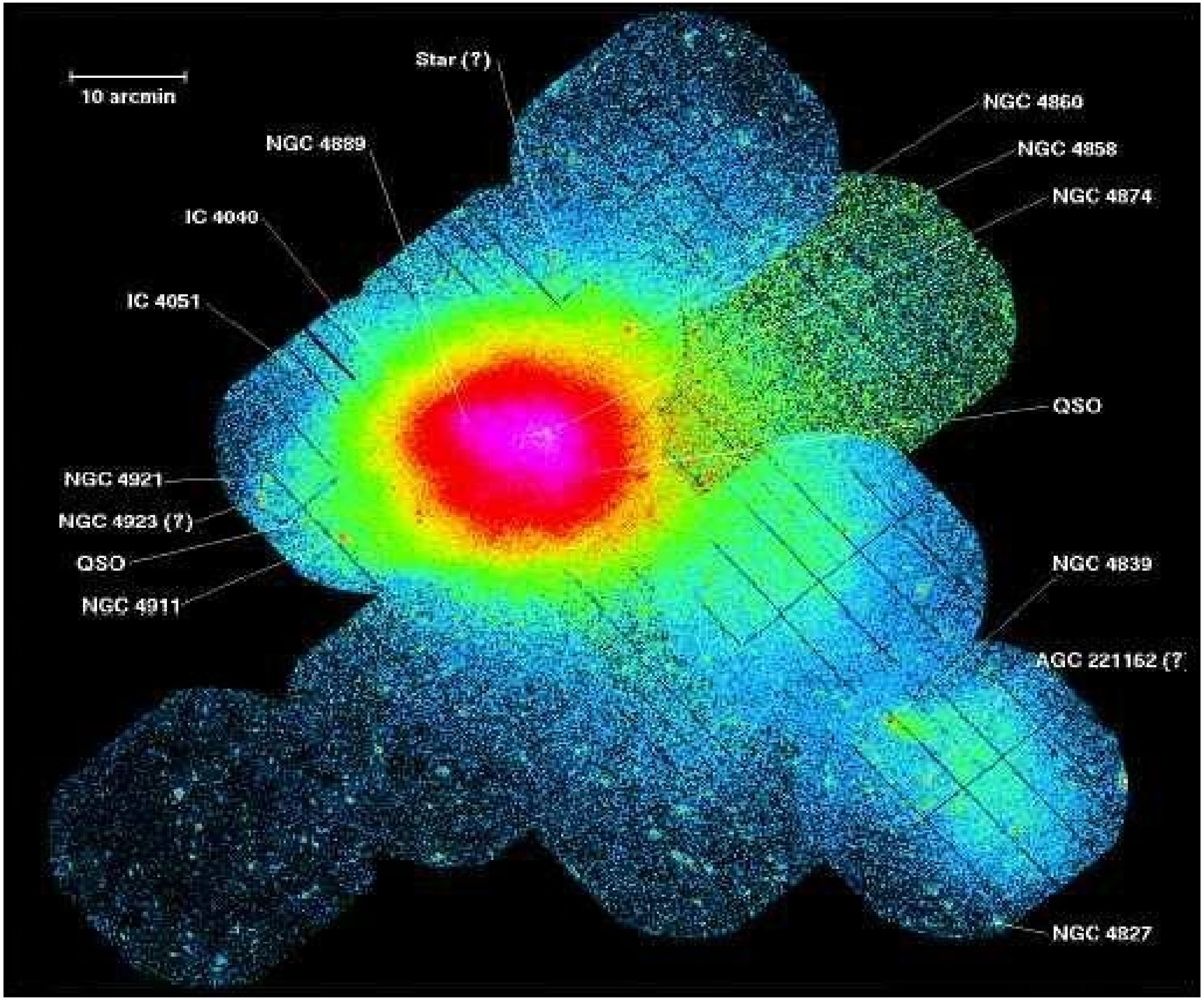
$$t_{br} = \frac{6\cdot10^{3}}{n_e \bar{g}_{ff}} T^{1/2} \quad yr$$

$$j_{br}(T) = 2.4\cdot10^{-27}\, T^{1/2} n_e^2 \bar{g}_{ff} \quad erg\, s^{-1}\, cm^{-3}$$

$$L_{br} = j_{br}\cdot V \quad erg\, s^{-1}$$

A cluster of galaxies:

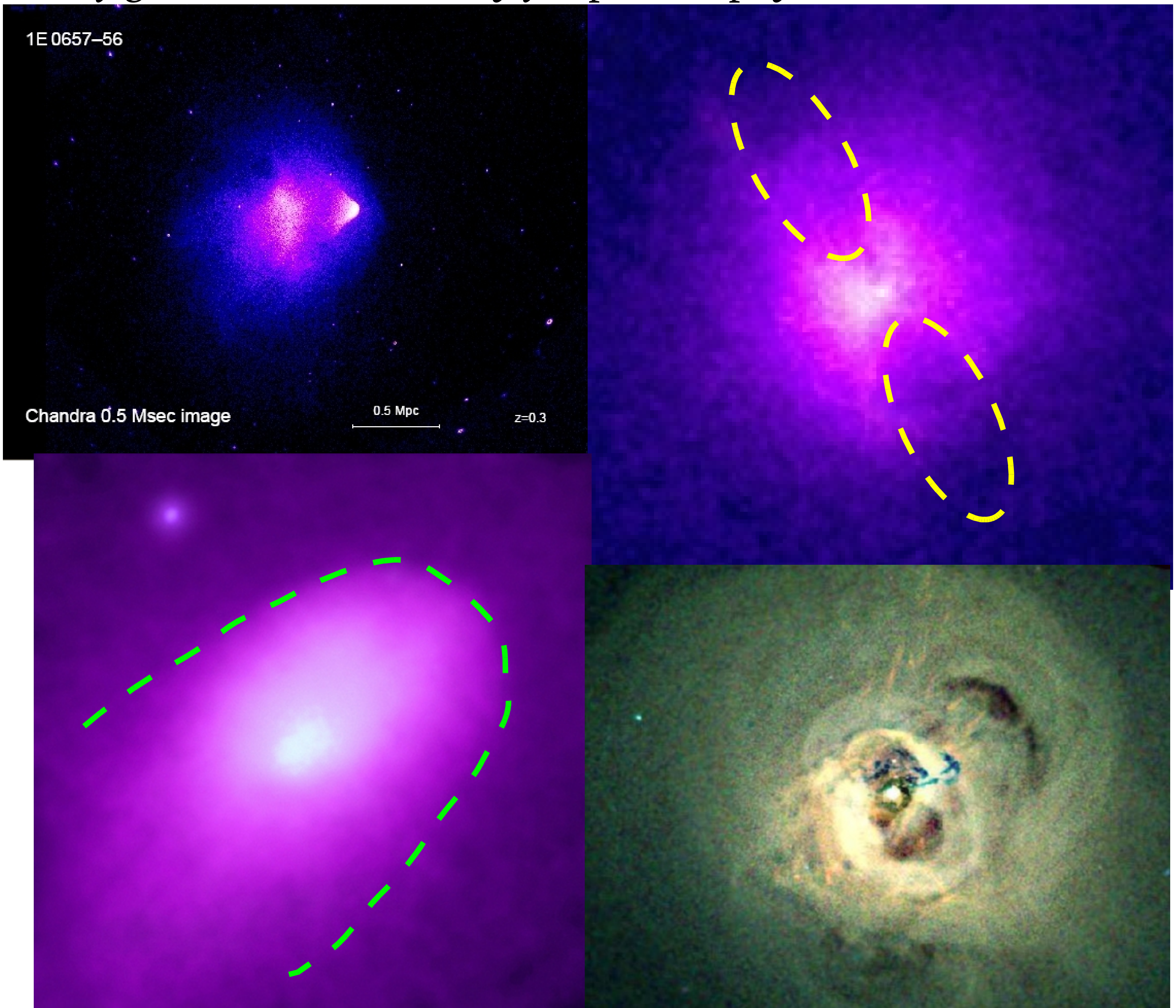
Coma ($z=0.0232$), size $\sim$ 1 Mpc



Typical values of IC hot gas have radiative cooling time exceeding 10 Gyr

all galaxy clusters are X-ray emitters

Cluster of galaxies: a laboratory for plasma physics



Thermal free-free absorption: *energy gained by a free moving electron.*
At thermal equilibrium (Kirchhoff 's law)

$$S(\nu) = \frac{j(\nu,T)}{\mu(\nu,T)} = B(\nu,T) = 2\frac{h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

$$j(\nu,T) = \mu(\nu,T)B(\nu,T) \qquad [\text{Kirchhoff's law}]$$

for an isotropically emitting plasma cloud via thermal bremsstrahlung

$$\frac{dW}{dt\,dV\,d\nu} = 4\pi J_{br}(\nu,T)$$

and therefore we can get

$$\mu_{br}(\nu,T) = \frac{J_{br}(\nu,T)}{4\pi B(\nu,T)} =$$

$$= \frac{1}{4\pi}\,6.8\cdot 10^{-38}\,T^{-1/2} n_e n_Z Z^2 \bar{g}_{ff}(\nu,T)\frac{c^2}{2h\nu^3}(1-e^{-h\nu/kT})$$

$$\Rightarrow \quad \mu_{br}(\nu,T) \approx n_e^2 T^{-1/2}\nu^{-3}(1-e^{-h\nu/kT})$$

strong dependency on frequency

– *at high frequencies $\mu(\nu,T)$ is negligible (ν^{-3} and $e^{-h\nu/kT}$)*

– *at low frequencies ($h\nu \ll kT$)*

$$\mu_{br}(\nu,T) \approx n_e^2 T^{-1/2} \nu^{-3} (1 - e^{-h\nu/kT})$$

$$\approx n_e^2 T^{-1/2} \nu^{-3} \left(1 - 1 + \frac{h\nu}{kT}\right) = n_e^2 \; T^{-3/2} \nu^{-2}$$

$$\mu_{br}(\nu,T) = \frac{4e^6}{3 m_e h c} \left(\frac{2\pi}{3 k m_e}\right)^{1/2} n_e n_Z Z^2 T^{-3/2} \nu^{-2} \bar{g}_{ff}(T)$$

$\bar{g}_{ff}(T) \approx 10$; *therefore*

$$\mu_{br}(\nu,T) \approx 0.018\, T^{-3/2} \nu^{-2} n_e n_Z Z^2 g_{ff}(T) \simeq \boxed{0.2\, T^{-3/2} \nu^{-2} n_e n_Z Z^2}$$

The total brightness from a free-free emitting cloud: consider emission + absorption

$$B(\nu,T) = \frac{1}{4\pi}\frac{j_{br}(\nu,T)}{\mu_{br}(\nu,T)}(1-e^{-\tau(\nu,T)}) = B_{BB}(\nu,T)(1-e^{-\tau(\nu,T)})$$

according to radiative transfer

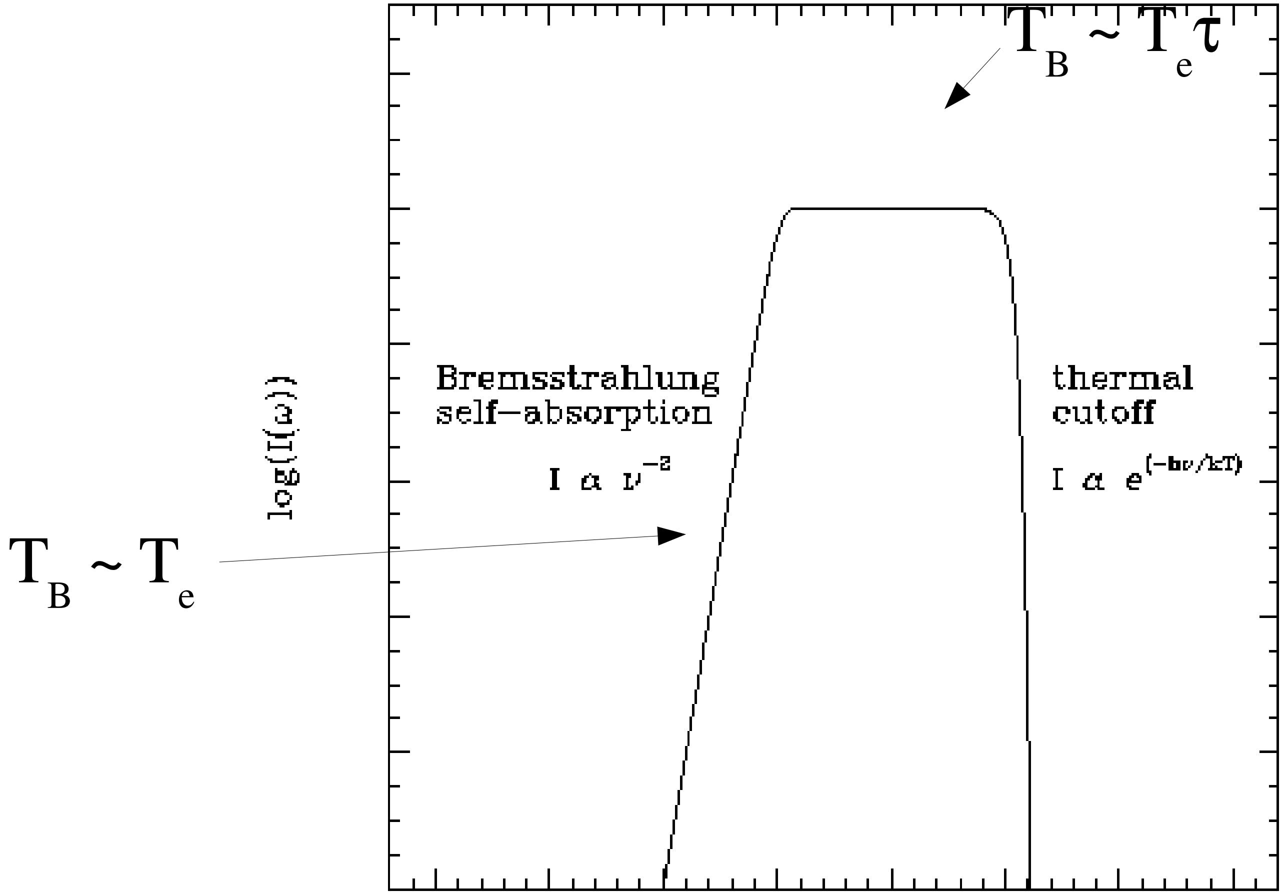
Therefore:

$$B(\nu,T) \approx \frac{T^{-1/2} n_e n_Z Z^2 \bar{g}_{ff}(\nu,T)}{T^{-3/2}\nu^{-2} n_e n_Z Z^2 \bar{g}_{ff}(\nu,T)}(1-e^{-\tau(\nu,T)}) \approx T\nu^2 (1-e^{-\tau(\nu,T)})$$

and the opacity [$\tau(\nu,T) = \mu(\nu,T)\, l$; where l = mean free path] determines the spectral shape and the regime:

1. *optically thick* $\tau(\nu,T) \gg 1$

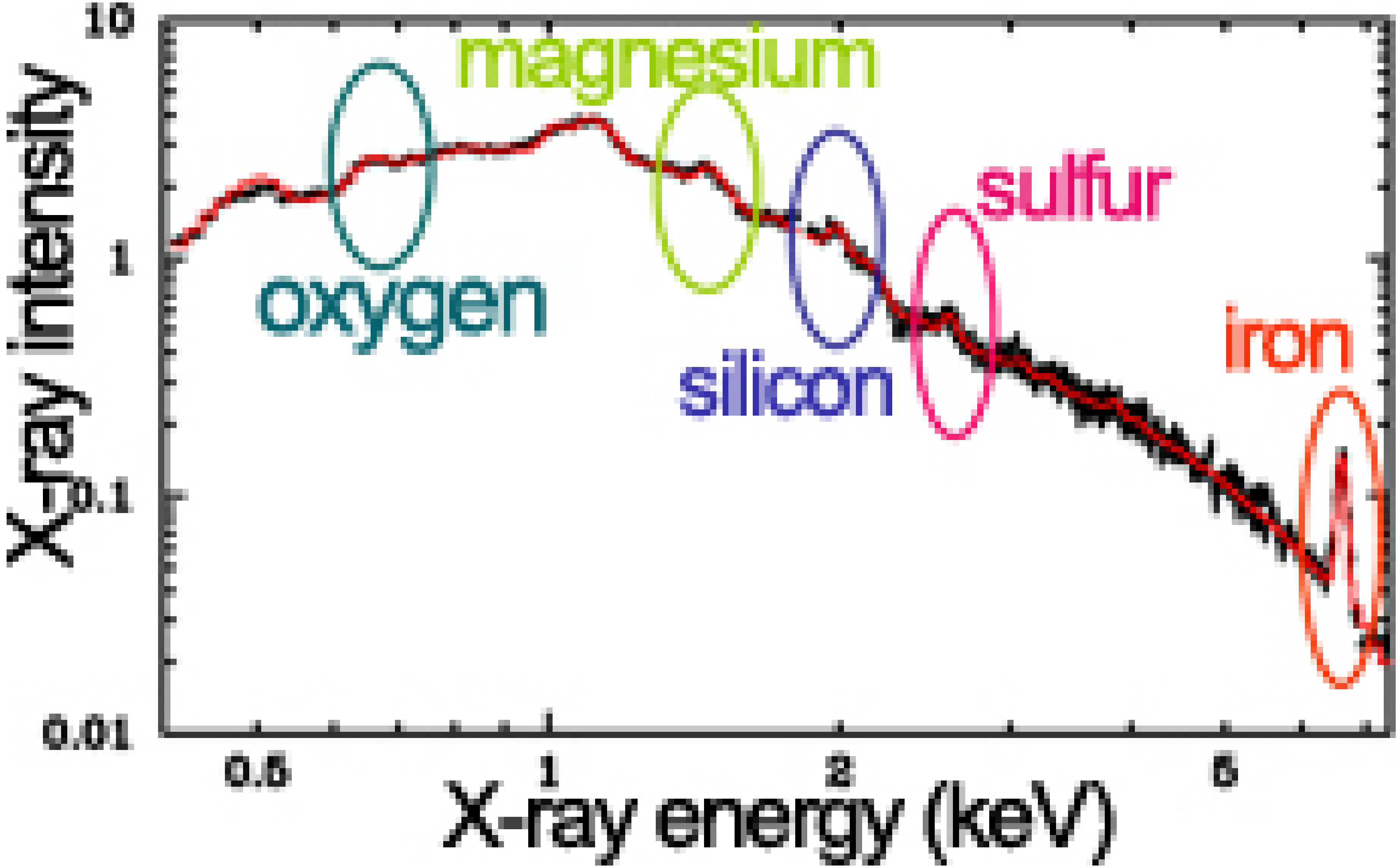
2. *optically thin* $\tau(\nu,T) \ll 1$



$$\tau \gg 1 \;\rightarrow\; \left(1 - e^{-\tau}\right) = \left(1 - \frac{1}{e^{-\tau}}\right) \approx 1 \qquad\qquad \rightarrow\; B(\nu, T) \approx T\nu^2$$

$$\tau \gg 1 \;\rightarrow\; \left(1 - e^{-\tau}\right) = \left|1 - 1 + \tau - o(\tau^2)\right| \approx \tau = \mu\, l_o \qquad \rightarrow\; B(\nu, T) \approx T\nu^2\tau$$

$$\mu(\nu) = \frac{J_{[br]}(\nu)}{4\pi B_{BB}(\nu, T)} \quad \text{where } B_{BB} \text{ in } R-J \text{ approximation} \approx \frac{T\nu^2}{(2h\nu^3/c^2)\cdot(kT/h\nu)} \approx T^{-1/2}$$


10
1
0.1
0.01
X-ray intensity
0.5
1
2
5
X-ray energy (keV)
magnesium
oxygen
silicon
sulfur
iron

Relativistic bremsstrahlung

minor remark: the Gaunt Factor is slightly different wrt the case of "thermal br.'

The emissivity for relativistic bremsstrahlung as a function of v *for a given velocity* v *is:*

$$j_{rel,br}(v,\nu) = \frac{32\pi}{3}\frac{e^6}{m_e^2 c^3}\frac{1}{v} n_e n_Z Z^2 \ln\left(\frac{183}{Z^{1/3}}\right)$$

N.B. T is not a relevant concept any more, v ***must be used***

The typical velocities of the particles are now relativistic and the ***energy (velocity) distribution is described by a power law****:*

$$n_e(E) \approx n_{e,o} E^{-\delta}$$

Integrating over velocities (energies)

$$j_{rel,br}(\nu) \approx \int_{h\nu}^{\infty} n_e(E) n_Z Z^2 dE \approx \int_{h\nu}^{\infty} E^{-\delta} dE \simeq \frac{E^{-\delta+1}}{1-\delta} \approx \nu^{-\delta+1}$$

the emitted spectrum is a power – law !

solar wind:

$n_e \sim 4 - 7 \quad cm^{-3}$

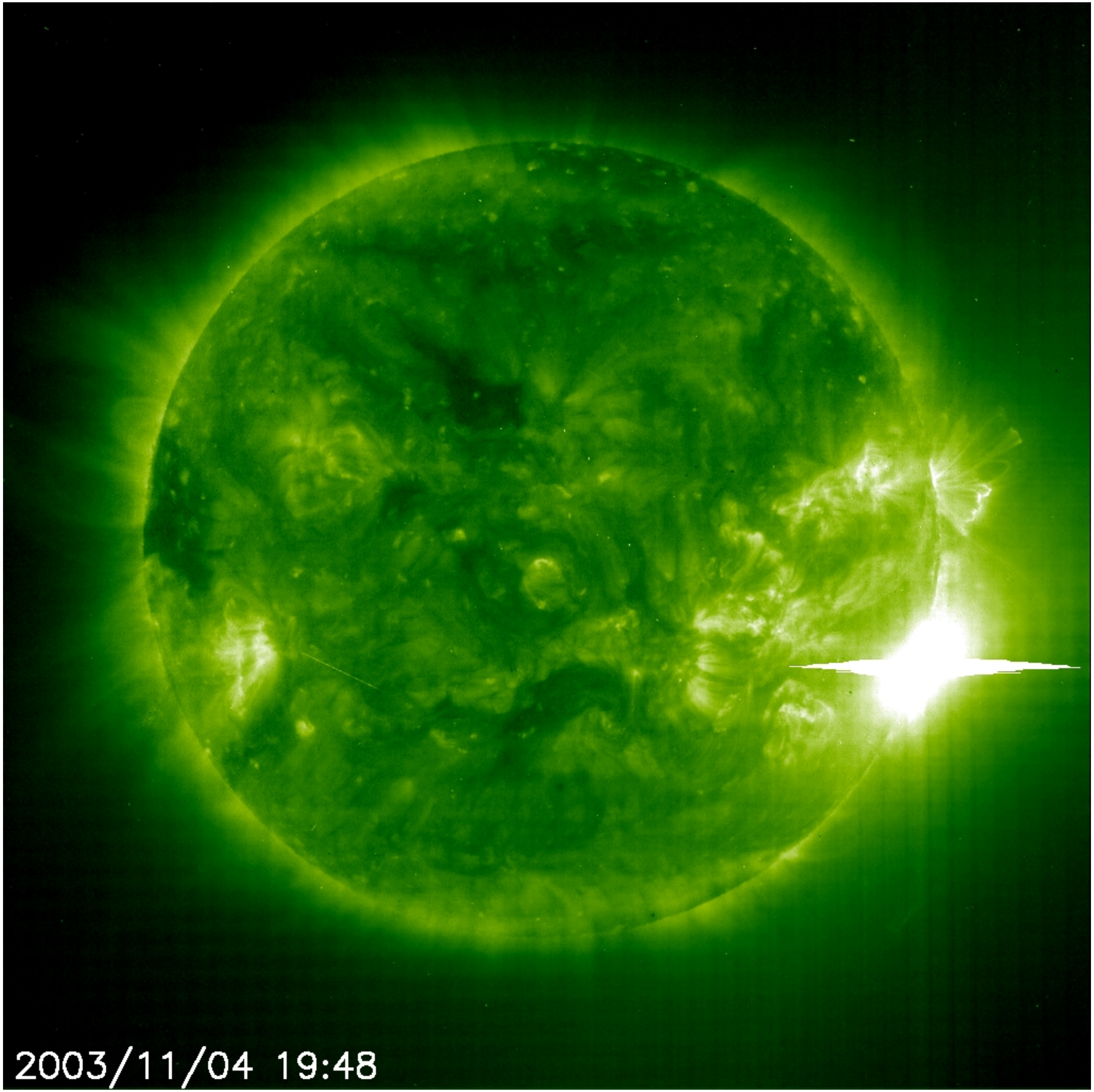
$v_{swind} \sim 300 - 900 \ km\ s^{-1}$

T ~ 150 000 K

(not consistent with v!)

flares:

relativistic particles with power–law energy distribution



Final remarks on bremsstrahlung emission:

a photon is emitted when a moving electron deeply penetrates the Coulomb field of positive ion and then is decelerated

the energy of the photon can never exceed the kinetic energy of the electron

generally the "classical" description of the phenomenon is adequate, the quantum mechanics requires an appropriate Gaunt Factor only

the individual interaction produces polarized radiation (fixed direction of E and B fields of the e-m wave wrt the acceleration)

all the interactions produced in a hot plasma originate unpolarized emission (random orientation of the plane of interaction)

we have either "thermal" or "relativistic" bremsstrahlung, depending on the population (velocity/energy distribution) of the electrons.